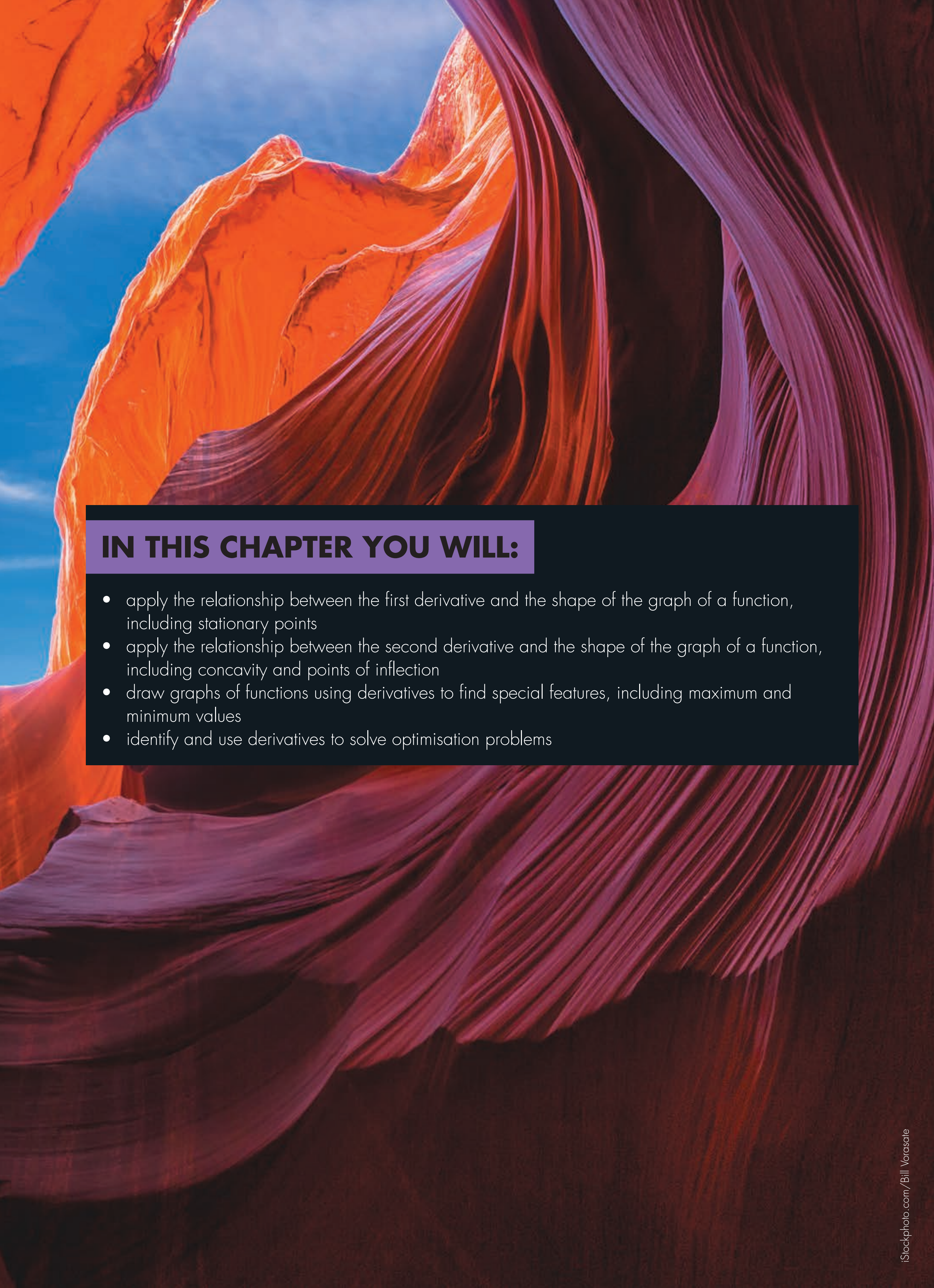
CALCULUS

6.

GEOMETRICAL APPLICATIONS OF DIFFERENTIATION

We can use first and second derivatives to find the shape of functions, including special features such as stationary points, and draw their graphs. We will also use differentiation to solve practical optimisation problems.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- apply the relationship between the first derivative and the shape of the graph of a function, including stationary points
- apply the relationship between the second derivative and the shape of the graph of a function, including concavity and points of inflection
- draw graphs of functions using derivatives to find special features, including maximum and minimum values
- identify and use derivatives to solve optimisation problems

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TERMINOLOGY

concavity: The shape of a curve as it bends; it can be concave up or concave down.

global maximum or minimum: The absolute highest or lowest value of a function over a given domain.

horizontal point of inflection: A stationary point where the concavity of the curve changes.

local maximum or minimum: a relatively high or low value of a function shown graphically as a turning point.

maximum point: A stationary point where the curve reaches a peak.

minimum point: A stationary point where the curve reaches a trough.

monotonic increasing or **decreasing:** A function that is always increasing or decreasing.

point of inflection: A point at which the curve is neither concave upwards nor downwards, but where the concavity changes.

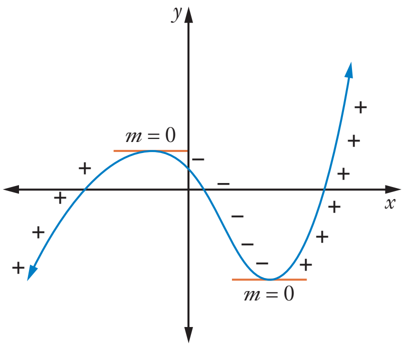
stationary point: A point on the graph of $y = f(x)$ where the tangent is horizontal and its gradient $f'(x) = 0$. It could be a maximum point, minimum point or a horizontal point of inflection.

WS

The sign of the derivative

6.01 Increasing and decreasing curves

You have already seen how the derivative describes the shape of a curve.



Sign of the first derivative

If $f'(x) > 0$, the graph of $y = f(x)$ is increasing.

If $f'(x) < 0$, the graph of $y = f(x)$ is decreasing.

If $f'(x) = 0$, the graph of $y = f(x)$ has a **stationary point**.

Sometimes a curve is **monotonic increasing** or **decreasing** (*always* increasing or decreasing).

Monotonic increasing or decreasing functions

A curve is monotonic increasing if $f'(x) > 0$ for all x.

A curve is monotonic decreasing if $f'(x) < 0$ for all x.

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EXAMPLE 1

a Find all x values for which the curve $f(x) = x^2 - 4x + 1$ is increasing.

b Find any stationary points on the curve $y = x^3 - 48x - 7$.

Solution

a $f'(x) = 2x - 4$

For increasing curve:

$f'(x) > 0$

$2x - 4 > 0$

$2x > 4$

$x > 2$

So the curve is increasing for $x > 2$.

b $y' = 3x^2 - 48$

For stationary points:

$y' = 0$

$3x^2 - 48 = 0$

$x^2 - 16 = 0$

$x^2 = 16$

$x = \pm 4$

When $x = 4$:

$y = 4^3 - 48(4) - 7$

$= -135$

When $x = -4$:

$y = (-4)^3 - 48(-4) - 7$

$= 121$

So the stationary points are $(4, -135)$ and $(-4, 121)$.

Exercise 6.01 Increasing and decreasing curves

1 For what x values is the function $f(x) = -2x^2 + 8x - 1$ increasing?

2 Find all values of x for which the curve $y = 2x^2 - x$ is decreasing.

3 Find the domain over which the function $f(x) = 4 - x^2$ is increasing.

4 Find values of x for which the curve $y = x^2 - 3x - 4$ is:

a decreasing **b** increasing **c** stationary

5 Show that the function $f(x) = -2x - 7$ is always (monotonic) decreasing.

6 Prove that $y = x^3$ is monotonic increasing for all $x \neq 0$.

7 Find the stationary point on the curve $f(x) = x^3$.

8 Find all x values for which the curve $y = 2x^3 + 3x^2 - 36x + 9$ is stationary.

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9 Find all stationary points on the curve:

a $y = x^2 - 2x - 3$ **b** $f(x) = 9 - x^2$

c $y = 2x^3 - 9x^2 + 12x - 4$ **d** $y = x^4 - 2x^2 + 1$

10 Find any stationary points on the curve $y = (x - 2)^4$.

11 EXT1 Find all values of x for which the curve $f(x) = x^3 - 3x + 4$ is decreasing.

12 EXT1 Find the domain over which the curve $y = x^3 + 12x^2 + 45x - 30$ is increasing.

13 Find any values of x for which the curve $y = 2x^3 - 21x^2 + 60x - 3$ is:

a stationary **b** EXT1 decreasing **c** EXT1 increasing

14 The function $f(x) = 2x^2 + px + 7$ has a stationary point at $x = 3$. Evaluate p.

15 Evaluate a and b if $y = x^3 - ax^2 + bx - 3$ has stationary points at $x = -1$ and $x = 2$.

16 **a** Find the derivative of $y = x^3 - 3x^2 + 27x - 3$.

b Show that the curve is monotonic increasing for all values of x.

17 Sketch a function with $f'(x) > 0$ for $x < 2$, $f'(2) = 0$ and $f'(x) < 0$ when $x > 2$.

18 Sketch a curve with $\frac{dy}{dx} < 0$ for $x < 4$, $\frac{dy}{dx} = 0$ when $x = 4$ and $\frac{dy}{dx} > 0$ for $x > 4$.

19 Sketch a curve with $\frac{dy}{dx} > 0$ for all $x \neq 1$ and $\frac{dy}{dx} = 0$ when $x = 1$.

20 Sketch a function that has $f'(x) > 0$ in the domain $(-\infty, -2) \cup (5, \infty)$, $f'(x) = 0$ for $x = -2$ and $x = 5$, and $f'(x) < 0$ in the domain $(-2, 5)$.

21 A function has $f(3) = 2$ and $f'(3) < 0$. Show this information on a sketch.

22 The derivative of a function is positive at the point $(-2, -1)$. Show this information on a graph.

23 Find the stationary points on the curve $y = (3x - 1)(x - 2)^4$.

24 Differentiate $y = x\sqrt{x+1}$. Hence find the stationary point on the curve, giving the exact coordinates.

25 The curve $f(x) = ax^4 - 2x^3 + 7x^2 - x + 5$ has a stationary point at $x = 1$. Find the value of a.

26 Show that $f(x) = \sqrt{x}$ has no stationary points.

27 Show that $f(x) = \frac{1}{x^3}$ has no stationary points.

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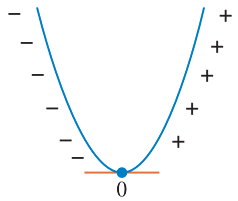
6.02 Stationary points

WS
Stationary points

In Year 11, Chapter 8, *Introduction to calculus*, you learned about 3 types of stationary points: minimum point, maximum point and horizontal point of inflection.

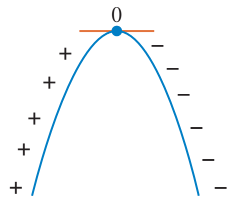
Minimum and maximum turning points

At a local **minimum point**, the curve is decreasing on the LHS and increasing on the RHS.



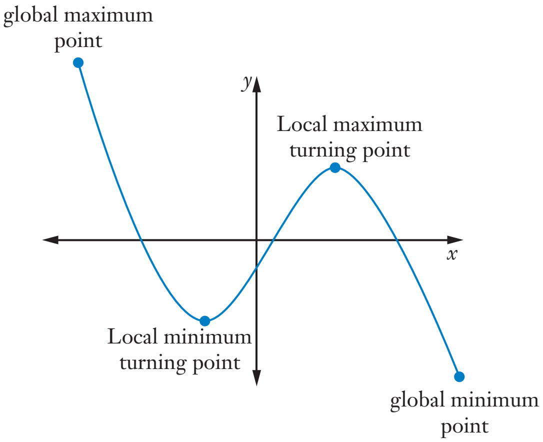
x	LHS	Minimum	RHS
$f'(x)$	< 0	0	> 0

At a local **maximum point**, the curve is increasing on the LHS and decreasing on the RHS.



x	LHS	Minimum	RHS
$f'(x)$	> 0	0	< 0

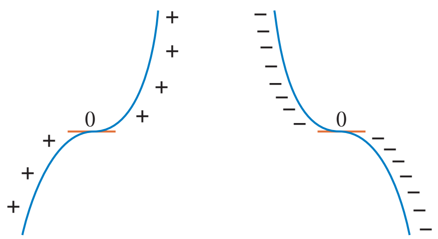
These stationary points are called **local maximum or minimum** points because they are not necessarily the **global maximum or minimum** points on the curve.



Horizontal point of inflection

These curves are increasing or decreasing on **both** sides of the horizontal **point of inflection**.
It is not a turning point since the curve does not turn around at this point.

We will learn more about points of inflection in the next section.



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EXAMPLE 2

Find any stationary points on the curve $f(x) = 2x^3 - 15x^2 + 24x - 7$ and determine their nature.

'determine their nature' means find what type of stationary point they are

Solution

$f'(x) = 6x^2 - 30x + 24$

For stationary points:

$$\begin{aligned} f'(x) &= 0 \\ 6x^2 - 30x + 24 &= 0 \\ 6(x^2 - 5x + 4) &= 0 \\ x^2 - 5x + 4 &= 0 \\ (x - 1)(x - 4) &= 0 \\ x = 1, \quad x &= 4 \end{aligned}$$

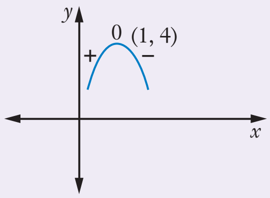
So there are 2 stationary points, where $x = 1$ and $x = 4$.

$$\begin{aligned} f(1) &= 2(1)^3 - 15(1)^2 + 24(1) - 7 \\ &= 4 \end{aligned}$$

So (1, 4) is a stationary point.

To determine its nature, choose a point close to (1, 4) on the LHS and RHS, for example, $x = 0$ and $x = 2$, and test the sign of $f'(x)$.

x	0	1	2
$f'(x)$	24	0	−12
	+		−



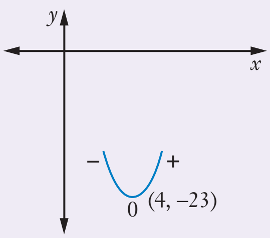
Positive to negative, so (1, 4) is a maximum point.

$$\begin{aligned} f(4) &= 2(4)^3 - 15(4)^2 + 24(4) - 7 \\ &= -23 \end{aligned}$$

So (4, −23) is a stationary point.

To determine its nature, choose, for example, $x = 2$ and $x = 5$.

x	2	4	5
$f'(x)$	−12	0	24
	−		+



Negative to positive, so (4, −23) is a minimum point.

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Exercise 6.02 Stationary points

1 Find the stationary point on the curve $y = x^2 - 1$ and show that it is a minimum point.

2 Find the stationary point on the curve $y = x^4$ and determine its type.

3 The function $f(x) = 7 - 4x - x^2$ has one stationary point. Find its coordinates and show that it is a maximum turning point.

4 Find the turning point on the curve $y = 3x^2 + 6x + 1$ and determine its nature.

5 For the curve $y = (4 - x)^2$, find the turning point and determine its nature.

6 The curve $y = x^3 - 6x^2 + 5$ has 2 turning points. Find them and use the derivative to determine their nature.

7 Find the turning points on the curve $y = x^3 - 3x^2 + 5$ and determine their nature.

8 Find any stationary points on the curve $f(x) = x^4 - 2x^2 - 3$. What type of stationary points are they?

9 The curve $y = x^3 - 3x + 2$ has 2 stationary points. Find their coordinates and determine their type.

10 The curve $y = x^5 + mx^3 - 2x^2 + 5$ has a stationary point at $x = -1$. Find the value of m.

11 For a certain function, $f'(x) = 3 + x$. For what value of x does the function have a stationary point? What type of stationary point is it?

12 A curve has $f'(x) = x(x + 1)$. For what x values does the curve have stationary points? What type are they?

13 **a** Differentiate $P = 2x + \dfrac{50}{x}$ with respect to x.

b Find any stationary points on the curve and determine their nature.

14 For the function $A = \dfrac{h^2 - 2h + 5}{8}$, find any stationary points and determine their nature.

15 Find any stationary points on the function $V = 40r - \pi r^3$ correct to 2 decimal places, and determine their nature.

16 Find any stationary points on the curve $S = 2\pi r + \dfrac{120}{r}$ correct to 2 decimal places, and determine their nature.

17 **a** Differentiate $A = x\sqrt{3600 - x^2}$.

b Find any stationary points on $A = x\sqrt{3600 - x^2}$ (to 1 decimal place) and determine their nature.

6.03 Concavity and points of inflection

Concavity

Shapes of curves

The first derivative $f'(x)$ is the rate of change of the function $y = f(x)$.

Similarly, the second derivative $f''(x)$ is the rate of change of the first derivative $f'(x)$.

This means the relationship between $f''(x)$ and $f'(x)$ is the same as the relationship between $f'(x)$ and $f(x)$.

Relationship between 1st and 2nd derivatives

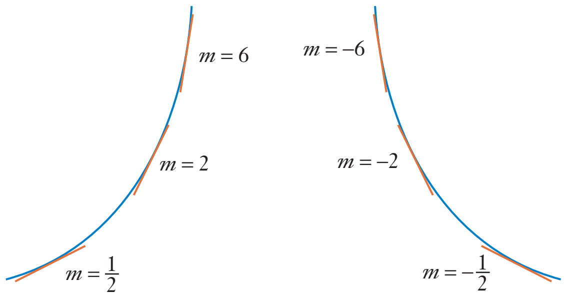
If $f''(x) > 0$ then $f'(x)$ is increasing.

If $f''(x) < 0$ then $f'(x)$ is decreasing.

If $f''(x) = 0$ then $f'(x)$ is stationary.

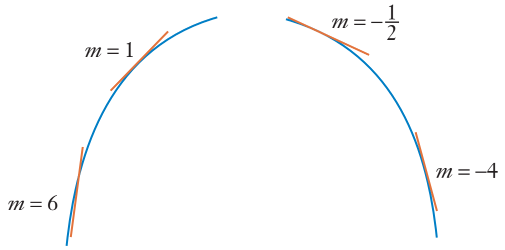
The sign of the second derivative shows the shape of the graph.

If $f''(x) > 0$ then $f'(x)$ is increasing. This means that the gradient of the tangent is increasing.



Notice the upward shape of these curves. The curve lies above the tangents. We say that the curve is **concave upwards**.

If $f''(x) < 0$ then $f'(x)$ is decreasing. This means that the gradient of the tangent is decreasing.



Notice the downward shape of these curves. The curve lies below the tangents. We say that the curve is **concave downwards**.

Sign of 2nd derivative

If $f''(x) > 0$, the curve is concave upwards.

If $f''(x) < 0$, the curve is concave downwards.

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EXAMPLE 3

Find the domain over which the curve $f(x) = 2x^3 - 7x^2 - 5x + 4$ is concave downwards.

Solution

$f'(x) = 6x^2 - 14x - 5$

$f''(x) = 12x - 14$

For concave downwards:

$f''(x) < 0$

$12x - 14 < 0$

$12x < 14$

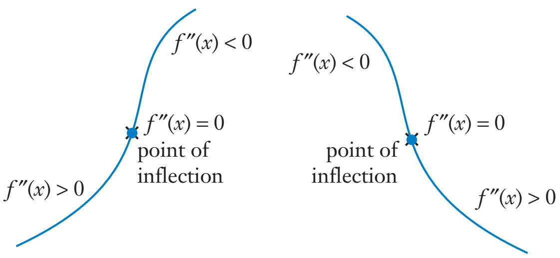
$x < \frac{14}{12}$

$x < 1\frac{1}{6}$

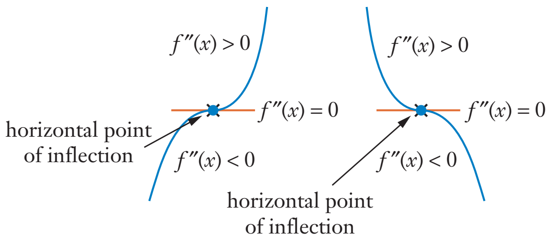
So the domain over which the curve is concave downwards is $(-\infty, 1\frac{1}{6})$

Points of inflection

At the point where $f''(x) = 0$, $f'(x)$ is constant. This means that the gradient of the tangent is neither increasing nor decreasing. This happens when the curve goes from being concave upwards to concave downwards, or concave downwards to concave upwards. We say that the curve is changing **concavity** at a **point of inflection**. The diagrams above show a point of inflection and the change in concavity as the curve changes shape.



The diagrams on the right show a **horizontal point of inflection** that occurs at a stationary point. Notice that the tangent is horizontal at the point of inflection.



Points of inflection

If $f''(x) = 0$, and concavity changes, it is a **point of inflection**.

If $f'(x) = 0$ also, it is a **horizontal point of inflection.**

EXAMPLE 4

a Find the point of inflection on the curve $y = x^3 - 6x^2 + 5x + 9$.

b Does the function $y = x^4$ have a point of inflection?

Solution

a $y' = 3x^2 - 12x + 5$

$y'' = 6x - 12$

For point of inflection, $y'' = 0$ and concavity changes.

$6x - 12 = 0$

$x = 2$

Check that concavity changes by choosing values on the LHS and RHS, for example $x = 1$ and $x = 3$, and testing the sign of the second derivative y''.

x	1	2	3
y''	-6	0	6
	$-$		$+$

Since concavity changes (negative to positive), there is a point of inflection at $x = 2$.

When $x = 2$:

$y = 2^3 - 6(2)^2 + 5(2) + 9$

$= 3$

So (2, 3) is a point of inflection.

b $\frac{dy}{dx} = 4x^3$

$\frac{d^2y}{dx^2} = 12x^2$

For point of inflection, $\frac{d^2y}{dx^2} = 0$ and concavity changes.

$12x^2 = 0$

$x^2 = 0$

$x = 0$

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Check that concavity changes by choosing values on the LHS and RHS, for example, $x = \pm 1$, and test the sign of $\frac{d^2y}{dx^2}$.

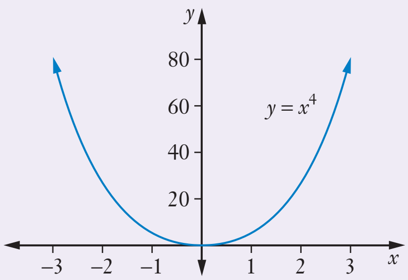
x	-1	0	1
$\frac{d^2y}{dx^2}$	12	0	12
	+		+

Since concavity doesn't change (both sides are positive), (0, 0) is not a point of inflection.

So $y = x^4$ does not have a point of inflection.

We can see this by drawing the graph of $y = x^4$.

This graph has a turning point at (0, 0).



Exercise 6.03 Concavity and points of inflection

1 For what values of x is the curve $y = x^3 + x^2 - 2x - 1$ concave upwards?

2 Find all values of x for which the function $f(x) = (x - 3)^3$ is concave downwards.

3 Prove that the curve $y = 8 - 6x - 4x^2$ is always concave downwards.

4 Show that the curve $y = x^2$ is always concave upwards.

5 Find the domain over which the curve $f(x) = x^3 - 7x^2 + 1$ is concave downwards.

6 Find any points of inflection on the curve $g(x) = x^3 - 3x^2 + 2x + 9$.

7 Find the points of inflection on the curve $y = x^4 - 6x^2 + 12x - 24$.

8 Find the stationary point on the curve $y = x^3 - 2$ and show that it is a point of inflection.

9 EXT1 Find all values of x for which the function $f(x) = x^4 + 2x^3 - 12x^2 + 12x - 1$ is concave downwards.

10 Determine whether there are any points of inflection on the curve:

a $y = x^6$ **b** $y = x^7$ **c** $y = x^5$

d $y = x^9$ **e** $y = x^{12}$

11 Sketch a curve that is always concave up.

12 Sketch a curve where $f''(x) < 0$ for $x > 1$ and $f''(x) > 0$ for $x < 1$.

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13 Find any points of inflection on the curve $y = x^4 - 8x^3 + 24x^2 - 4x - 9$.

14 Show that $f(x) = \dfrac{2}{x^2}$ is concave upwards for all $x \neq 0$.

15 For the function $f(x) = 3x^5 - 10x^3 + 7$:

a Find any points of inflection.

b Find which of these points are horizontal points of inflection (stationary points).

16 **a** Show that the curve $y = x^4 + 12x^2 - 20x + 3$ has no points of inflection.

b Describe the concavity of the curve.

17 If $y = ax^3 - 12x^2 + 3x - 5$ has a point of inflection at $x = 2$, evaluate a.

18 Evaluate p if $f(x) = x^4 - 6px^2 - 20x + 11$ has a point of inflection at $x = -2$.

19 The curve $y = 2ax^4 + 4bx^3 - 72x^2 + 4x - 3$ has points of inflection at $x = 2$ and $x = -1$. Find the values of a and b.

20 The curve $y = x^6 - 3x^5 + 21x - 8$ has 2 points of inflection.

a Find these points of inflection.

b Show that they are not stationary points.

WS

The shape of a curve

6.04 Interpreting rates of change graphically

We can find out more about the shape of a graph if we combine the results from the first and second derivatives.

EXAMPLE 5

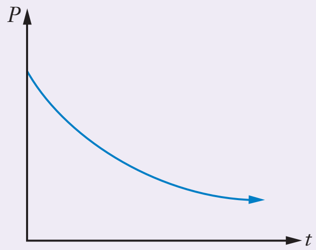
a For a particular curve, $f(2) = -1$, $f'(2) > 0$ and $f''(2) < 0$. Draw the shape of the curve at this point.

b The curve below shows the population (P) of unemployed people over time t months.

i Describe the sign of $\dfrac{dP}{dt}$ and $\dfrac{d^2P}{dt^2}$.

ii How is the population of unemployed people changing over time?

iii Is the rate of change of unemployment increasing or decreasing?



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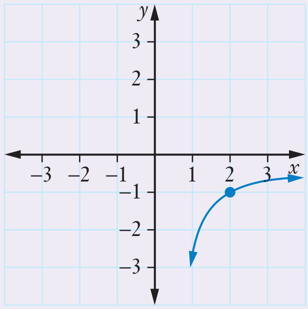
Solution

a $f(2) = -1$ means that the point $(2, -1)$ lies on the curve.

If $f'(2) > 0$, the curve is increasing at this point.

If $f''(2) < 0$, the curve is concave downwards at this point.



b **i** The curve is decreasing, so $\frac{dP}{dt} < 0$.

The curve is concave upwards, so $\frac{d^2P}{dt^2} > 0$.

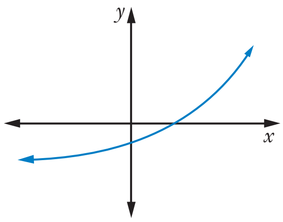
ii Since the curve is decreasing, the number of unemployed people is decreasing.

iii Since the curve is concave upwards, the (negative) gradient is increasing. This means that the rate of change of unemployment is increasing.

Exercise 6.04 Interpreting rates of change graphically

1 For each curve, describe the sign of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

a



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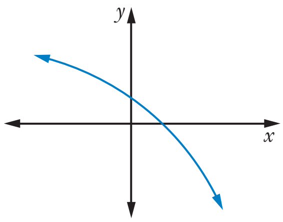

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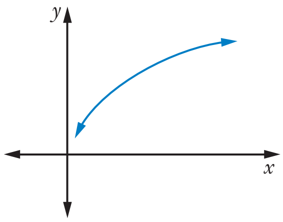

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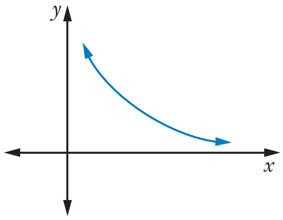

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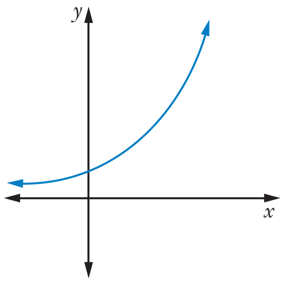

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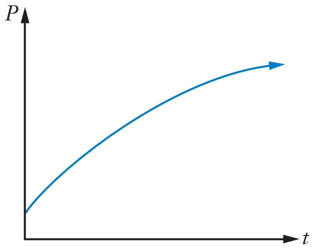
2 The curve below shows the population of a colony of sea lions.



a Describe the sign of the first and second derivatives.

b Is the rate of change of the sea lion population increasing?

Photo courtesy Margaret Grove

3 Inflation is increasing, but the rate of increase is slowing. Draw a graph to show this trend.

4 Draw a sketch to show the shape of each curve.

a $f'(x) < 0$ and $f''(x) < 0$

b $f'(x) > 0$ and $f''(x) < 0$

c $f'(x) < 0$ and $f''(x) > 0$

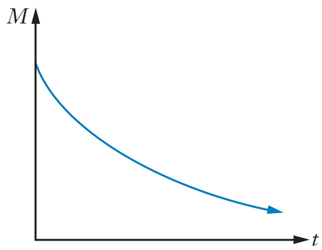
d $f'(x) > 0$ and $f''(x) > 0$

5 The size of classes at a local TAFE college is decreasing and the rate at which this is happening is decreasing. Draw a graph to show this.

6 As an iceblock melts, the rate at which it melts increases. Draw a graph to show this information.

7 The graph shows the decay of a radioactive substance.

Describe the sign of $\dfrac{dM}{dt}$ and $\dfrac{d^2M}{dt^2}$.



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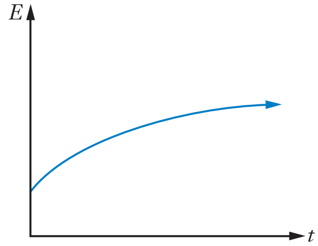
8 The population P of fish in a certain lake was studied over time. At the start of the study the number of fish was 2500.

a During the study, $\frac{dP}{dt} < 0$. What does this say about the number of fish during the study?

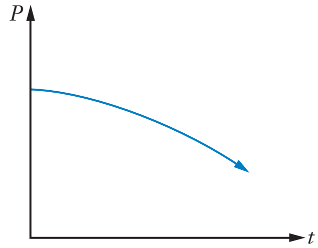
b If at the same time, $\frac{d^2P}{dt^2} > 0$, what can you say about the population rate of change?

c Sketch the graph of the population P against t.

9 The graph shows the level of education of youths in a certain rural area over the past 100 years. Describe how the level of education has changed over this period of time. Include mention of the rate of change.



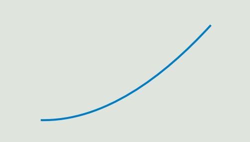
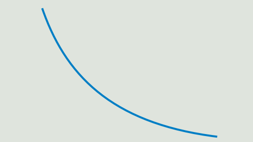
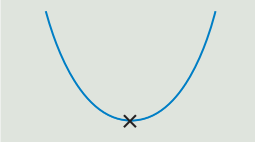
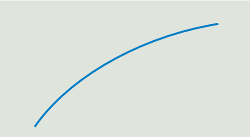
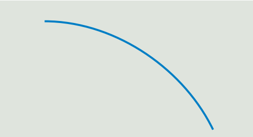
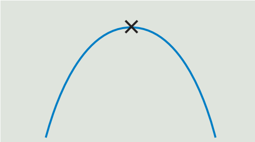
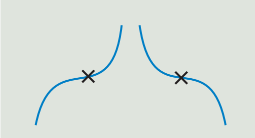
10 The graph shows the number of students in a high school over several years. Describe how the school population is changing over time, including the rate of change.



6.05 Stationary points and the second derivative

Putting the first and second derivatives together gives this summary of the shape of a curve.

Shape of a curve and the derivatives

	$\frac{dy}{dx} > 0$	$\frac{dy}{dx} < 0$	$\frac{dy}{dx} = 0$
$\frac{d^2y}{dx^2} > 0$			
$\frac{d^2y}{dx^2} < 0$			
$\frac{d^2y}{dx^2} = 0$			

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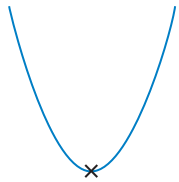
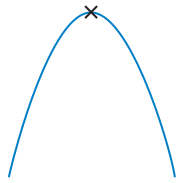
We can use the table to find the requirements for stationary points.

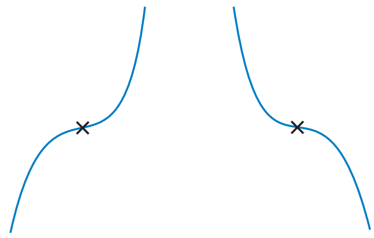
If $f'(x) = 0$ and $f''(x) > 0$, there is a minimum turning point (concave upwards).



If $f'(x) = 0$ and $f''(x) < 0$, there is a maximum turning point (concave downwards).



If $f'(x) = 0$ and $f''(x) = 0$ and concavity changes, then there is a horizontal point of inflection.



Stationary points and the derivatives

Minimum turning point: $f'(x) = 0$ and $f''(x) > 0$

Maximum turning point: $f'(x) = 0$ and $f''(x) < 0$

Horizontal point of inflection: $f'(x) = 0$, $f''(x) = 0$ and concavity changes

Now we can use the second derivative to determine the nature of stationary points.

EXAMPLE 6

a Find the stationary points on the curve $f(x) = 2x^3 - 3x^2 - 12x + 7$ and distinguish between them.

b Find the stationary point on the curve $y = 2x^5 - 3$ and determine its nature.

Solution

a $f'(x) = 6x^2 - 6x - 12$

$f''(x) = 12x - 6$

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For stationary points:

$$f'(x) = 0$$
$$6x^2 - 6x - 12 = 0$$
$$x^2 - x - 2 = 0$$
$$(x+1)(x-2) = 0$$
$$x = -1, x = 2$$

$$f(-1) = 2(-1)^3 - 3(-1)^2 - 12(-1) + 7$$
$$= 14$$
$$f''(-1) = 12(-1) - 6$$
$$= -18$$
$$< 0 \text{ (concave downwards)}$$

So $(-1, 14)$ is a maximum turning point.

$$f(2) = 2(2)^3 - 3(2)^2 - 12(2) + 7$$
$$= -13$$
$$f''(2) = 12(2) - 6$$
$$= 18$$
$$> 0 \text{ (concave upwards)}$$

So $(2, -13)$ is a minimum turning point.

b $y' = 10x^4$

$$y'' = 40x^3$$

For stationary points:

$$y' = 0$$
$$10x^4 = 0$$
$$x^4 = 0$$
$$x = 0$$

When $x = 0$:

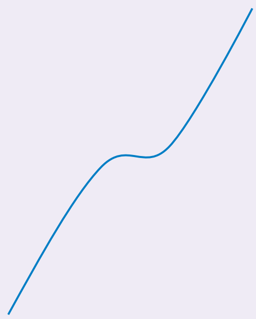
$$y = 2(0)^5 - 3$$
$$= -3$$
$$y'' = 40(0)^3$$
$$= 0$$

Check that concavity changes by choosing values on the LHS and RHS, for example, $x = \pm 1$.

x	-1	0	1
y''	-40	0	40
	$-$		$+$

Since concavity changes, $(0, -3)$ is a horizontal point of inflection.

The table also tells us that the curve changes from concave downwards to concave upwards.



Exercise 6.05 Stationary points and the second derivative

1 Find the stationary point on the curve $y = x^2 - 2x + 1$ and determine its nature.

2 Find the stationary point on the curve $f(x) = 3x^4 + 1$ and determine what type of point it is.

3 Find the stationary point on the curve $y = 3x^2 - 12x + 7$ and show that it is a minimum turning point.

4 Determine the stationary point on $y = x - x^2$ and show that it is a maximum point.

5 Show that $f(x) = 2x^3 - 5$ has a horizontal point of inflection and find its coordinates.

6 Does the function $f(x) = 2x^5 + 3$ have a stationary point? If it does, determine its nature.

7 Find any stationary points on $f(x) = 2x^3 + 15x^2 + 36x - 50$ and determine their nature.

8 Find the stationary points on the curve $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ and determine whether they are maximum or minimum points.

9 Find any stationary points on the curve $y = (4x^2 - 1)^4$ and determine their nature.

10 **a** Find any stationary points on the curve $y = 2x^3 - 27x^2 + 120x$ and distinguish between them.

b Find any points of inflection on the curve.

11 Find any stationary points on the curve $y = (x - 3)\sqrt{4 - x}$ and determine their nature.

12 Find any stationary points on the curve $f(x) = x^4 + 8x^3 + 16x^2 - 1$ and determine their nature.

13 The curve $y = ax^2 - 4x + 1$ has a stationary point where $x = \frac{1}{2}$.

a Find the value of a.

b Hence, or otherwise, find the stationary point and determine its nature.

14 The curve $y = x^3 - mx^2 + 5x - 7$ has a stationary point where $x = -1$. Find the value of m.

15 The curve $y = ax^3 + bx^2 - x + 5$ has a point of inflection at $(1, -2)$.
Find the values of a and b.

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6.06 Curve sketching

We can sketch the graph of a function by using special features such as intercepts, stationary points and points of inflection. Here is a summary of strategies for sketching a curve.

WS Curve sketching

WS Further curve sketching

WS Curve sketching

WS Curve sketching with derivatives

WS Further graphs

Curve sketching

Curve sketching

Sketching curves

- Find stationary points $\left(\frac{dy}{dx}=0\right)$, and determine their nature.
- Find points of inflection $\left(\frac{d^2y}{dx^2}=0\right)$, and check that concavity changes.
- Find any x-intercepts ($y = 0$), and y-intercepts ($x = 0$).
- Find domain and range.
- Find any asymptotes or other discontinuities.
- Find limiting behaviour of the function.
- Use the symmetry of the function where possible:
  - check if the function is even: $f(-x) = f(x)$
  - check if the function is odd: $f(-x) = -f(x)$

EXAMPLE 7

a Find any stationary points and points of inflection on the curve $f(x) = x^3 - 3x^2 - 9x + 1$ and hence sketch the curve.

b Sketch the curve of the composite function $y = f(g(x))$ where $f(x) = 2x + 1$ and $g(x) = x^3$, showing any important features.

Solution

a $f'(x) = 3x^2 - 6x - 9$

$f''(x) = 6x - 6$

For stationary points:

$$f'(x) = 0$$

$$3x^2 - 6x - 9 = 0$$

$$x^2 - 2x - 3 = 0$$

$$(x-3)(x+1) = 0$$

$$x = 3, \qquad x = -1$$

$$f(3) = 3^3 - 3(3)^2 - 9(3) + 1$$

$$= -26$$

$$f''(3) = 6(3) - 6$$

$$= 12$$

$$> 0 \quad \text{(concave upwards)}$$

So $(3, -26)$ is a minimum turning point.

$f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 1$

$\quad = 6$

$f''(-1) = 6(-1) - 6$

$\quad = -12$

$\quad < 0$ (concave downwards)

So $(-1, 6)$ is a maximum turning point.

For points of inflection:

$f''(x) = 0$

$6x - 6 = 0$

$6x = 6$

$x = 1$

Check concavity changes by choosing values on LHS and RHS e.g. $x = 0$ and $x = 2$.

x	0	1	2
$f''(x)$	−6	0	6

Since concavity changes, $x = 1$ is at a point of inflection.

$f(1) = 1^3 - 3(1)^2 - 9(1) + 1$

$\quad = -10$

So $(1, -10)$ is a point of inflection.

For x-intercept, $y = 0$:

$0 = x^3 - 3x^2 - 9x + 1$

This has no factors so we can't find the x-intercepts.

For y-intercept, $x = 0$:

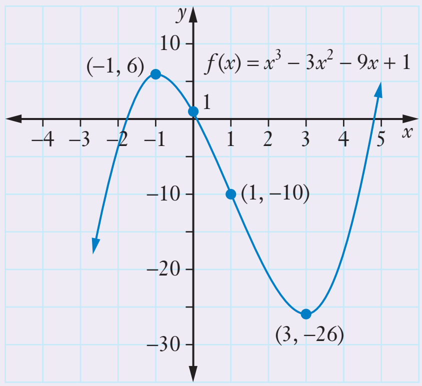
$f(0) = 0^3 - 3(0)^2 - 9(0) + 1$

$\quad = 1$

$f(x) = x^3 - 3x^2 - 9x + 1$ is a cubic function with no symmetry or discontinuities.

It is not an even or odd function.

Notice that the point of inflection at $(1, -10)$ is not a stationary point. It is the point where the graph naturally changes concavity.



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b $y = f(g(x))$

$= 2x^3 + 1$

$\frac{dy}{dx} = 6x^2$

$\frac{d^2y}{dx^2} = 12x$

For stationary points:

$\frac{dy}{dx} = 0$

$6x^2 = 0$

$x^2 = 0$

$x = 0$

When $x = 0$:

$y = 2(0)^3 + 1$

$= 1$

$\frac{d^2y}{dx^2} = 12(0)$

$= 0$

Check concavity either side:

x	–1	0	1
$\frac{d^2y}{dx^2}$	–12	0	12

Since concavity changes, (0, 1) is a horizontal point of inflection.

For x-intercepts, $y = 0$

$0 = 2x^3 + 1$

$-1 = 2x^3$

$-0.5 = x^3$

$\sqrt[3]{-0.5} = x$

$-0.8 \approx x$

For y-intercept, $x = 0$

$y = 2(0)^3 + 1$

$= 1$

This is (0, 1), the point of inflection.

This is a cubic function. We can make the graph more accurate by finding some extra points.

When $x = -1$:

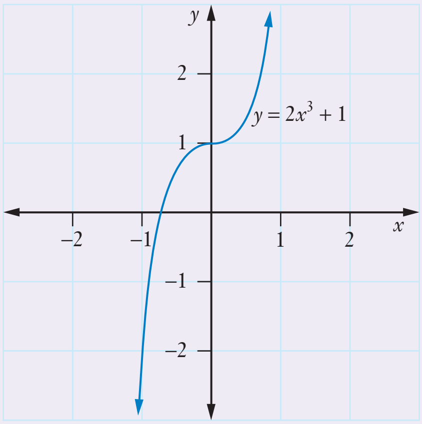
$y = 2(-1)^3 + 1$

$= -1$

When $x = 1$:

$y = 2(1)^3 + 1$

$= 3$



You can use derivatives to help sketch other functions, for example trigonometric, exponential and logarithmic graphs.

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EXAMPLE 8

a Sketch the curve $y = xe^x$, showing any important features.

b EXT1 Given $f(x) = x(x - 2)^2$, sketch the graph of $y^2 = f(x)$.

Solution

a
$$y = xe^x$$
$$y' = u'v + v'u \text{ where } u = x \text{ and } v = e^x$$
$$u' = 1 \qquad v' = e^x$$
$$y' = 1 \times e^x + e^x \times x$$
$$= e^x(1 + x)$$
$$y'' = u'v + v'u \text{ where } u = e^x \text{ and } v = 1 + x$$
$$u' = e^x \qquad v' = 1$$
$$y'' = e^x \times (1 + x) + 1 \times e^x$$
$$= e^x(2 + x)$$

For stationary points:
$$y' = 0$$
$$e^x(1 + x) = 0$$
$$1 + x = 0 \qquad (e^x \neq 0)$$
$$x = -1$$

When $x = -1$:
$$y = -1e^{-1}$$
$$= -\frac{1}{e}$$
$$y'' = e^{-1}(2 + -1)$$
$$= \frac{1}{e}$$
$$> 0 \qquad \text{(concave upwards)}$$

So $\left(-1, -\frac{1}{e}\right)$ is a minimum turning point.

For x-intercepts, $y = 0$:
$$0 = xe^x$$
$$x = 0 \qquad (e^x \neq 0)$$

For y-intercepts, $x = 0$:
$$y = 0e^0$$
$$= 0$$

The general exponential function $y = a^x$ has an asymptote at the x-axis.

Limiting behaviour as $x \to \pm\infty$:

As $x \to \infty$, $xe^x \to \infty$ since x and e^x are both becoming large as x becomes large.

As $x \to -\infty$, $x \to -\infty$ but $e^x \to 0$ when x is negative $\left(\text{since } e^{-x} = \frac{1}{e^x}\right)$.

So $xe^x \to 0^-$ (it approaches zero from the negative side).

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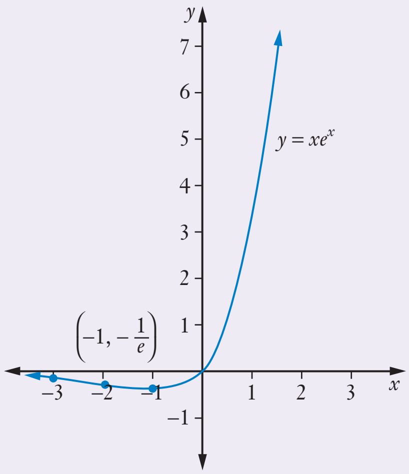
We can sketch this information on a graph.



b EXT1 $y^2 = f(x)$

So $\quad y^2 = x(x-2)^2$

$\quad\quad y = \pm\sqrt{x(x-2)^2}$

For $y = \sqrt{x(x-2)^2}$,

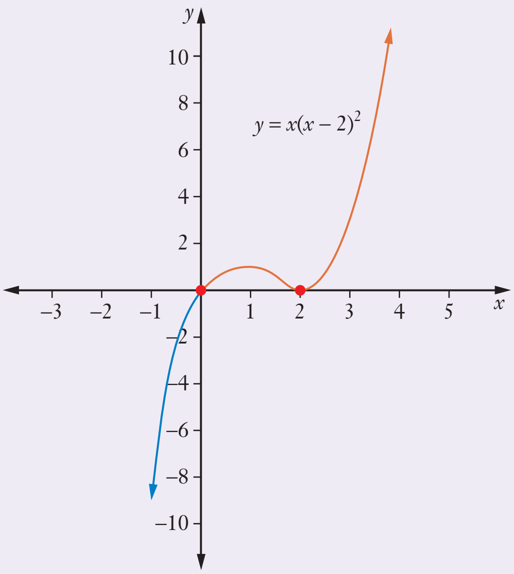
we need $x(x-2)^2 \geq 0$

First, sketch the cubic function $y = x(x-2)^2$ and find the parts of the graph on and above the x-axis.

For x-intercepts, $y = 0$:

$0 = x(x-2)^2$

$x = 0, x = 2$



This cubic function has a positive leading term, x-intercepts at 0 and 2.
$x = 2$ is a double root (multiplicity 2) so it is a turning point, as shown on the graph.

So $x(x-2)^2 \geq 0$ when $x \geq 0$.

Domain: $[0, \infty)$, Range: $[0, \infty)$ as shown by the brown section of the curve.

$$y = \sqrt{x(x-2)^2}$$
$$= \sqrt{x(x^2 - 4x + 4)}$$
$$= \sqrt{x^3 - 4x^2 + 4x}$$
$$= (x^3 - 4x^2 + 4x)^{\frac{1}{2}}$$
$$\frac{dy}{dx} = \frac{1}{2}(3x^2 - 8x + 4)\,(x^3 - 4x^2 + 4x)^{-\frac{1}{2}}$$
$$= \frac{3x^2 - 8x + 4}{2\sqrt{x^3 - 4x^2 + 4x}}$$
$$= \frac{3x^2 - 8x + 4}{2\sqrt{x(x-2)^2}} \qquad (x \neq 0, 2)$$

For stationary points,

$$\frac{dy}{dx} = 0$$
$$\frac{3x^2 - 8x + 4}{2\sqrt{x^3 - 4x^2 + 4x}} = 0$$
$$3x^2 - 8x + 4 = 0$$
$$(3x - 2)(x - 2) = 0$$
$$3x - 2 = 0, \quad x - 2 = 0$$
$$x = \frac{2}{3} \qquad x = 2 \text{ (not a stationary point since } x \neq 2)$$

When $x = \frac{2}{3}$:

$$y = \sqrt{\frac{2}{3}\left(\frac{2}{3} - 2\right)^2}$$
$$\approx 1.089$$

So $\left(\frac{2}{3}, 1.089\right)$ is a stationary point.

The second derivative is difficult to find, so we can check the first derivative:

x	0.6	$\frac{2}{3}$	0.7
$\frac{dy}{dx}$	0.13	0	–0.06
	+		–

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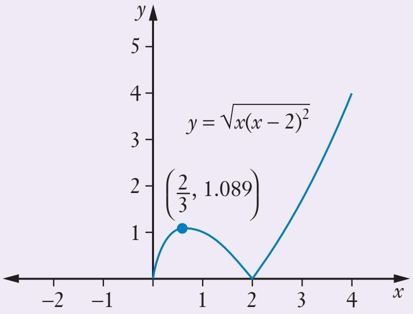
Positive to negative, so $\left(\frac{2}{3}, 1.089\right)$ is a maximum turning point.

Now, using the brown section of the graph on p. 265, you need to calculate the square root of the y values of that graph to draw the new graph. The x-intercepts at 0 and 2 will be the same because $\sqrt{0} = 0$.

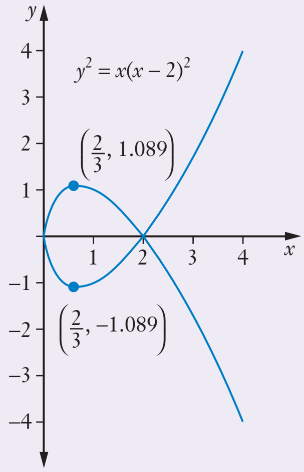
Drawing this information gives the graph below.

A table of values may help.



The graph of $y = -\sqrt{x(x-2)^2}$ is a reflection in the x-axis.

Putting both graphs together gives us the graph of $y^2 = x(x-2)^2$.



Notice that the derivative is undefined at $x = 0$ (where the tangent is vertical) and $x = 2$ (where there are 2 tangents).

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When the second derivative is hard to find, we can use the first derivative to check the type of stationary points.

EXAMPLE 9

Find any stationary points and sketch the function $y = x\sqrt{16 - x^2}$.

Solution

$$y = x\sqrt{16 - x^2}$$

$$y' = u'v + v'u \qquad u = x \qquad v = \sqrt{16 - x^2} = (16 - x^2)^{\frac{1}{2}}$$

$$u' = 1 \qquad v' = -2x \times \frac{1}{2}(16 - x^2)^{-\frac{1}{2}}$$

$$= -\frac{x}{\sqrt{16 - x^2}}$$

$$y' = 1 \times \sqrt{16 - x^2} + \left(-\frac{x}{\sqrt{16 - x^2}}\right) \times x$$

$$= \sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}}$$

For stationary points:

$$y' = 0$$

$$\sqrt{16 - x^2} - \frac{x^2}{\sqrt{16 - x^2}} = 0$$

$$16 - x^2 - x^2 = 0 \qquad \text{(multiplying both sides by } \sqrt{16 - x^2}\text{)}$$

$$16 - 2x^2 = 0$$

$$16 = 2x^2$$

$$8 = x^2$$

$$\pm\sqrt{8} = x$$

When $x = \sqrt{8}$:

$$y = \sqrt{8} \times \sqrt{16 - (\sqrt{8})^2}$$

$$= \sqrt{8} \times \sqrt{8}$$

$$= 8$$

So $(\sqrt{8}, 8)$ is a stationary point.

When $x = -\sqrt{8}$:

$$y = -\sqrt{8} \times \sqrt{16 - (-\sqrt{8})^2}$$

$$= -\sqrt{8} \times \sqrt{8}$$

$$= -8$$

So $(-\sqrt{8}, -8)$ is a stationary point.

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Since the second derivative is hard to find, we can check the first derivative on LHS and RHS of $\pm\sqrt{8} \approx 2.8$, to see where the curve is increasing and decreasing.

x	2	2.8	3
y'	+2.3	0	−0.8

Positive to negative, so $(\sqrt{8}, 8)$ is a maximum turning point.

x	−3	−2.8	−2
y'	−0.8	0	+2.3

Negative to positive, so $(-\sqrt{8}, -8)$ is a minimum turning point.

For x-intercepts, $y = 0$:

$$0 = x\sqrt{16 - x^2}$$

$$x = 0, \sqrt{16 - x^2} = 0$$

$$16 - x^2 = 0$$

$$16 = x^2$$

$$\pm 4 = x$$

For y-intercept, $x = 0$:

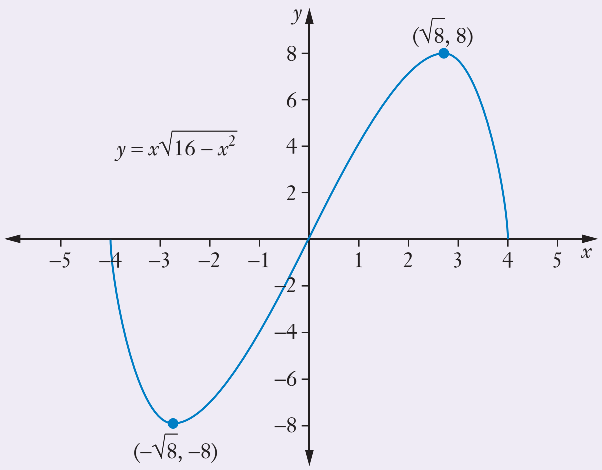
$$y = 0\sqrt{16 - 0^2}$$

$$= 0$$

Domain: $\sqrt{16 - x^2} \geq 0$

This simplifies to $-4 \leq x \leq 4$ or $[-4, 4]$ by solving the inequality or by noticing that the graph of $y = \sqrt{16 - x^2}$ is a semicircle with radius 4.

We can sketch this information on a graph.



Exercise 6.06 Curve sketching

1 Find the stationary point on the curve $f(x) = x^2 - 3x - 4$ and determine its type. Find the x- and y-intercepts of the graph of $f(x)$ and sketch the curve.

2 Sketch the graph of $y = 6 - 2x - x^2$, showing the stationary point.

3 Find the stationary point on the curve of the composite function $y = f(g(x))$ where $f(x) = x^3$ and $g(x) = x - 1$ and determine its nature. Hence sketch the curve.

4 Sketch the graph of $y = x^4 + 3$, showing any stationary points.

5 Find the stationary point on the curve $y = x^5$ and show that it is a point of inflection. Hence sketch the curve.

6 Sketch the graph of $f(x) = x^7$.

7 Find any stationary points on the curve $y = 2x^3 - 9x^2 - 24x + 30$ and sketch its graph.

8 **a** Determine any stationary points on the curve $y = x^3 + 6x^2 - 7$.

b Find any points of inflection on the curve.

c Sketch the curve.

9 Find any stationary points and points of inflection on the curve $y = f(x) + g(x)$ where $f(x) = x^3 - 7x^2 - 1$ and $g(x) = x^2 + 4$ and hence sketch the curve.

10 Find any stationary points and points of inflection on the curve $y = 2 + 9x - 3x^2 - x^3$. Hence sketch the curve.

11 Sketch the graph of $f(x) = 3x^4 + 4x^3 - 12x^2 - 1$, showing all stationary points.

12 EXT1 Find the stationary points on the curve $y = f(x)g(x)$ given $f(x) = (x - 4)$ and $g(x) = (x + 2)^2$, and hence sketch the curve.

13 Find all stationary points and points of inflection on the curve $y = (2x + 1)(x - 2)^4$. Sketch the curve.

14 Show that the curve $y = \dfrac{2}{1+x}$ has no stationary points. By considering the domain and range of the function, sketch the curve.

15 Sketch in the domain $[0, 2\pi]$, showing all stationary points:

a $y = \cos 2x$ **b** $y = 5 \sin 4x$

16 Draw the graph of each function, showing stationary points, points of inflection and other features.

a $y = x^2 \ln x$ **b** $y = \dfrac{x}{e^x}$ **c** $y = \dfrac{1}{x^2 - 1}$

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17 EXT1 Sketch the graph of each function, showing features such as stationary points, points of inflection or asymptotes.

a $y = \dfrac{1}{f(x)}$ where $f(x) = x^2 - 16$

b $y = \dfrac{f(x)}{g(x)}$ where $f(x) = x^2$ and $g(x) = x^2 + 4$

c $y = 1 + \dfrac{f(x)}{g(x)}$ given $f(x) = x$ and $g(x) = x^2 - 1$

d $y^2 = f(x)$ if $f(x) = (x - 1)(x - 2)(x + 3)$

6.07 Global maxima and minima

WS Greatest and least values

A curve may have local maximum and minimum turning points, but the absolute highest and lowest values of a function over a given domain are called the **global maximum or minimum values** of the function.

EXAMPLE 10

Find the global maximum and minimum values of y for the function $f(x) = x^4 - 2x^2 + 1$ in the domain $[-2, 3]$.

Solution

$f'(x) = 4x^3 - 4x$

$f''(x) = 12x^2 - 4$

For stationary points:

$$f'(x) = 0$$
$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$4x(x + 1)(x - 1) = 0$$
$$x = 0, \quad x = -1, \quad x = 1$$

$f(0) = 0^4 - 2(0)^2 + 1$
$= 1$

$f''(0) = 12(0)^2 - 4$
$= -4$
< 0 (concave downward)

So $(0, 1)$ is a maximum turning point.

$f(-1) = (-1)^4 - 2(-1)^2 + 1$
$= 0$

$f''(-1) = 12(-1)^2 - 4$
$= 8$
> 0 (concave upward)

So $(-1, 0)$ is a minimum turning point.

$f(1) = 1^4 - 2(1)^2 + 1$
$= 0$

$f''(1) = 12(1)^2 - 4$
$= 8$
> 0 (concave upward)

So $(1, 0)$ is a minimum turning point.

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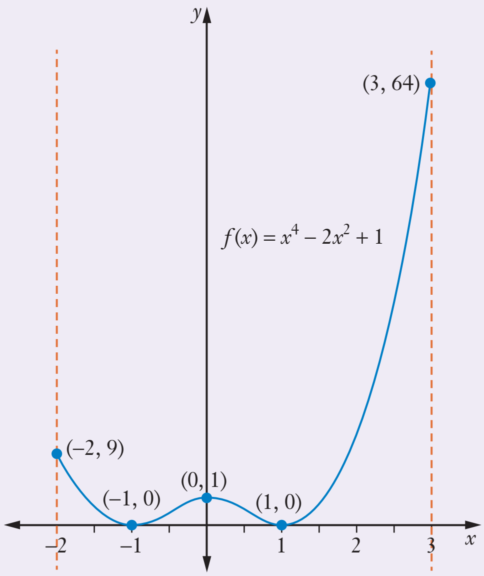
At the endpoints of the domain:

$$f(-2) = (-2)^4 - 2(-2)^2 + 1$$
$$= 9$$
$$f(3) = 3^4 - 2(3)^2 + 1$$
$$= 64$$

Checking, we also notice that $f(x) = x^4 - 2x^2 + 1$ is an even function.

$$f(-x) = (-x)^4 - 2(-x)^2 + 1$$
$$= x^4 - 2x^2 + 1$$
$$= f(x)$$

Drawing this information:



In the domain [–2, 3], the global maximum value is 64 and the global minimum value is 0.

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Exercise 6.07 Global maxima and minima

1 Sketch the graph of $y = x^2 + x - 2$ in the domain $[-2, 2]$ and find the maximum value of y in this domain.

2 Sketch the graph of $f(x) = 9 - x^2$ over the domain $[-4, 2]$. Hence find the maximum and minimum values of the function over this domain.

3 Find the maximum value of $y = x^2 - 4x + 4$ in the domain $[-3, 3]$.

4 Sketch the graph of $f(x) = 2x^3 + 3x^2 - 36x + 5$ for $-3 \le x \le 3$, showing any stationary points. Find the global maximum and minimum values of the function.

5 Find the global maximum for $y = x^5 - 3$ in the domain $[-2, 1]$.

6 Sketch the curve $f(x) = 3x^2 - 16x + 5$ for $0 \le x \le 4$ and find its global maximum and minimum.

7 Find the local and global maximum and minimum of $f(x) = 3x^4 + 4x^3 - 12x^2 - 3$ in the domain $[-2, 2]$.

8 Sketch $y = x^3 + 2$ over the domain $[-3, 3]$ and find its global minimum and maximum.

9 Sketch $y = \sqrt{x+5}$ for $-4 \le x \le 4$ and find its maximum and minimum values.

10 Show that $y = \dfrac{1}{x-2}$ has no stationary points. Find its maximum and minimum values in the domain $[-3, 3]$.

INVESTIGATION

THE LARGEST DISC

One disc 20 cm in diameter and one 10 cm in diameter are cut from a disc of cardboard 30 cm in diameter. Can you find the largest disc that can be cut from the remainder of the cardboard?

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6.08 Finding formulas for optimisation problems

Optimisation problems involve finding maximum or minimum values. For example, a salesperson wants to maximise profit; a warehouse manager wants to maximise storage; a driver wants to minimise petrol consumption; a farmer wants to maximise paddock size.

To solve an optimisation problem, we must first find a formula for the quantity that we are trying to maximise or minimise.

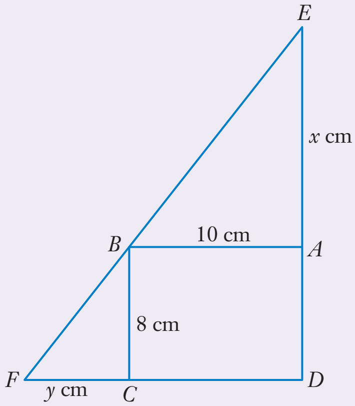
EXAMPLE 11

a A rectangular prism has a base with length twice its width. Its volume is 300 cm^3. Show that the surface area is given by $S = 4x^2 + \frac{900}{x}$.

b $ABCD$ is a rectangle with $AB = 10$ cm and $BC = 8$ cm. Length $AE = x$ cm and $CF = y$ cm.

i Show that $xy = 80$.

ii Show that triangle EDF has area given by $A = 80 + 5x + \frac{320}{x}$.



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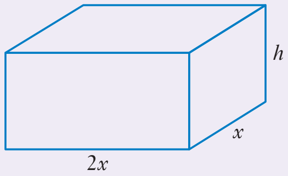
Solution

a Volume:

$$V = lwh$$
$$= 2x \times x \times h$$
$$= 2x^2h$$

$V = 300$:

$$300 = 2x^2h$$
$$\frac{300}{2x^2} = h \qquad [1]$$



Surface area:

$$S = 2(lw + wh + lh)$$
$$= 2(2x^2 + xh + 2xh)$$
$$= 2(2x^2 + 3xh)$$
$$= 4x^2 + 6xh$$

Substitute [1]:

$$S = 4x^2 + 6x \times \frac{300}{2x^2}$$
$$= 4x^2 + \frac{900}{x}$$

b **i** Triangles AEB and CBF are similar.

So $\frac{10}{y} = \frac{x}{8}$

$$xy = 80 \qquad [1]$$

ii Side $FD = y + 10$ and side $ED = x + 8$

Since $xy = 80$

$$y = \frac{80}{x} \qquad [2]$$

Area:

$$A = \frac{1}{2}bh$$
$$= \frac{1}{2}(y + 10)(x + 8)$$
$$= \frac{1}{2}(xy + 8y + 10x + 80)$$
$$= \frac{1}{2}\left(80 + 8 \times \frac{80}{x} + 10x + 80\right) \qquad \text{substituting [1] and [2]}$$
$$= \frac{1}{2}\left(160 + \frac{640}{x} + 10x\right)$$
$$= 80 + \frac{320}{x} + 5x$$

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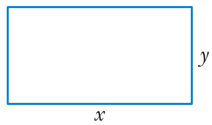
Exercise 6.08 Finding formulas for optimisation problems

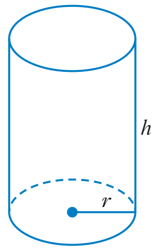
1 The area of a rectangle is to be 50 m^2. Show that its perimeter is given by the equation $P = 2x + \frac{100}{x}$.



2 A rectangular paddock on a farm is to have a fence with a 120 m perimeter. Show that the area of the paddock is given by $A = 60x - x^2$.

3 The product of 2 numbers is 20. Show that the sum of the numbers is $S = x + \frac{20}{x}$.

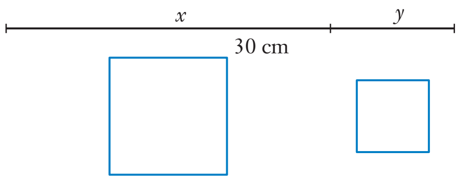
4 A closed cylinder is to have a volume of 400 cm^3. Show that its surface area is $S = 2\pi r^2 + \frac{800}{r}$.



5 A 30 cm length of wire is cut into 2 pieces and each piece bent to form a square as shown.

a Show that $y = 30 - x$.

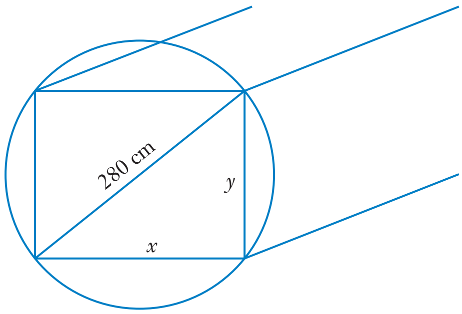
b Show that the total area of the 2 squares is given by $A = \frac{x^2 - 30x + 450}{8}$.



6 A timber post with a rectangular cross-sectional area is to be cut out of a log with a diameter of 280 mm as shown.

a Show that $y = \sqrt{78400 - x^2}$.

b Show that the cross-sectional area is given by $A = x\sqrt{78400 - x^2}$.



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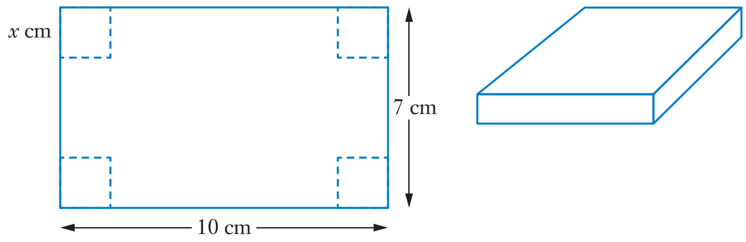
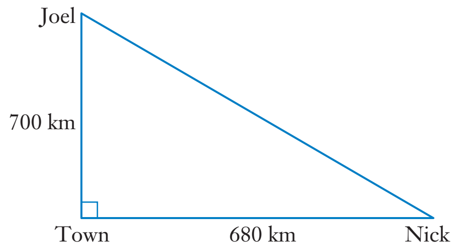
7 A 10 cm by 7 cm rectangular piece of cardboard has equal square corners with side x cm cut out. The sides are folded up to make an open box as shown. Show that the volume of the box is $V = 70x - 34x^2 + 4x^3$.

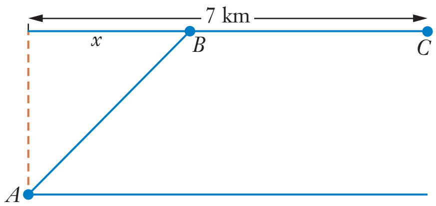


8 A travel agency calculates the expense E per person of organising a holiday in a group of x people as $E = 200 + 400x$. The cost C for each person taking a holiday is $C = 900 - 100x$. Show that the profit to the travel agency on a holiday with a group of x people is given by $P = 700x - 500x^2$.

9 Joel is 700 km north of a town, travelling towards it at an average speed of 75 km h^{-1}. Nick is 680 km east of the town, travelling towards it at 80 km h^{-1}.
Show that, after t hours, the distance between Joel and Nick is given by
$d = \sqrt{952400 - 213800t + 12025t^2}$.



10 Taylor swims from point A to point B across a 500 m wide river, then walks along the river bank to point C. The distance along the river bank is 7 km. If she swims at 5 km h^{-1} and walks at 4 km h^{-1}, show that the time taken to reach point C is given by $t = \dfrac{\sqrt{x^2 + 0.25}}{5} + \dfrac{7 - x}{4}$.



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6.09 Optimisation problems

WS Optimisation problems

WS Further optimisation problems

WS Starting maxima and minima problems

WS Applications of optimisation

Maximum volume

WS Applications of derivatives problems

WS Applications of derivatives assignment

WS Second derivative assignment

WS Second derivative problems

You can use derivatives to find the maximum or minimum value of a formula.

Always check that an answer gives a maximum or minimum value.

EXAMPLE 12

The equation for the expense per year, E (in units of \$10 000), of running a certain business is given by $E = x^2 - 6x + 12$, where x is the number (in 100s) of items manufactured.

a Find the expense of running the business if no items are manufactured.

b Find the number of items needed to minimise the expense of the business.

c Find the minimum expense of the business.

Solution

a When $x = 0$:

$E = 0^2 - 6(0) + 12$

$= 12$ (expense is in units of \$10 000)

So the expense of running the business when no items are manufactured is $12 \times \$10\,000 = \$120\,000$ per year.

b For stationary points:

$$\frac{dE}{dx} = 0$$

$$\frac{dE}{dx} = 2x - 6 = 0$$

$$2x = 6$$

$$x = 3 \quad (x \text{ is in 100s of items})$$

$$\frac{d^2E}{dx^2} = 2$$

$$> 0 \quad \text{(concave upwards)}$$

So $x = 3$ gives a minimum value.

$3 \times 100 = 300$

So 300 items manufactured each year will give the minimum expense.

c When $x = 3$:

$E = 3^2 - 6(3) + 12$

$= 3$

So the minimum expense per year is $3 \times \$10\,000 = \$30\,000$.

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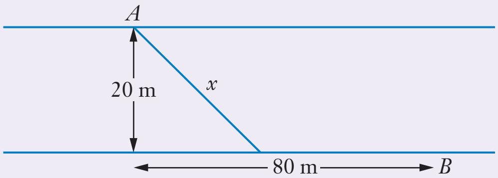
EXAMPLE 13

Optimisation problems

a The council wants to make a rectangular swimming area at the beach using the seashore on one side and a length of 300 m of shark-proof netting for the other 3 sides. What are the dimensions of the rectangle that encloses the greatest area?

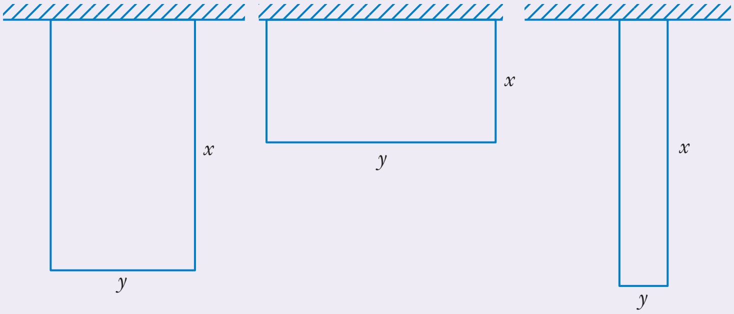
b Kristyn is at point A on one side of a 20 m wide river and needs to get to point B on the other side 80 m along the bank as shown. Kristyn swims to any point on the other bank and then runs along the side of the river to point B.



If she can swim at 7 km h^{-1} and run at 11 km h^{-1}, find x, the distance she swims to the nearest metre, to minimise her total travel time.

Solution

a Many different rectangles could have a perimeter of 300 m. Let the length of the rectangle be y and the width be x.



Perimeter: $2x + y = 300$ m

$y = 300 - 2x$ [1]

Area:

$A = xy$

$= x(300 - 2x)$ substituting [1]

$= 300\,x - 2x^2$

$$\frac{dA}{dx} = 300 - 4x$$

For stationary points:

$$\frac{dA}{dx} = 0$$

$300 - 4x = 0$

$300 = 4x$

$75 = x$

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$$\frac{d^2A}{dx^2} = -4$$

< 0 Concave downwards

So $x = 75$ gives maximum area.

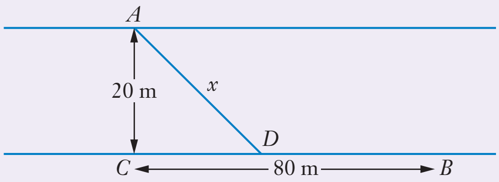
When $x = 75$: Substituting into [1]

$y = 300 - 2(75)$

$= 150$

So the dimensions that give the maximum area are 150 m × 75 m.

b First, we need to find a formula for the time Kristyn takes to run the distance $AD + DB$.



$AD = x$ so find DB:

$DB = 80 - CD$

By Pythagoras' theorem, $x^2 = 20^2 + CD^2$

$x^2 - 20^2 = CD^2$

$CD = \sqrt{x^2 - 400}$

$DB = 80 - \sqrt{x^2 - 400}$

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$s = \frac{d}{t}$$

$$st = d$$

$$t = \frac{d}{s}$$

Time taken to swim AD:

$$t_1 = \frac{x}{7}$$

Time taken to run DB:

$$t_2 = \frac{80 - \sqrt{x^2 - 400}}{11}$$

Total time:

$$t = t_1 + t_2$$

$$= \frac{x}{7} + \frac{80 - \sqrt{x^2 - 400}}{11}$$

$$= \frac{11x}{77} + \frac{7(80 - \sqrt{x^2 - 400})}{77}$$

$$= \frac{11x + 560 - 7(x^2 - 400)^{\frac{1}{2}}}{77}$$

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$$\frac{dt}{dx} = \frac{11 - 7 \times 2x \times \frac{1}{2}(x^2 - 400)^{-\frac{1}{2}}}{77}$$

$$= \frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77}$$

For minimum time: $\frac{dt}{dx} = 0$

$$\frac{11 - 7x(x^2 - 400)^{-\frac{1}{2}}}{77} = 0$$

$$11 - 7x(x^2 - 400)^{-\frac{1}{2}} = 0$$

$$11 = 7x(x^2 - 400)^{-\frac{1}{2}}$$

$$11 = \frac{7x}{\sqrt{x^2 - 400}}$$

$$11\sqrt{x^2 - 400} = 7x$$

$$121(x^2 - 400) = 49x^2 \qquad \text{squaring both sides}$$

$$121x^2 - 48\,400 = 49x^2$$

$$72x^2 = 48\,400$$

$$x^2 = 672.222\ldots$$

$$x = \sqrt{672.222\ldots.}$$

$$\approx 25.9$$

To check that t is a minimum:

x	25	25.9	26
$\frac{dt}{dx}$	−0.009	0	0.0006

Since the function is decreasing on LHS and increasing on RHS, t is a minimum at $x = 25.9$.

So Kristyn should swim a distance of 25.9 m to minimise her total travel time.

Exercise 6.09 Optimisation problems

1 The height, in metres, of a ball is given by the equation $h = 16t - 4t^2$, where t is time in seconds. Find when the ball will reach its maximum height, and what the maximum height will be.

2 The cost per hour of a bike ride is given by the formula $C = x^2 - 15x + 70$, where x is the distance travelled in km. Find the distance that gives the minimum cost.

3 The perimeter of a rectangle is 60 m and its length is x m.

a Show that the area of the rectangle is given by the equation $A = 30x - x^2$.

b Hence find the maximum area of the rectangle.

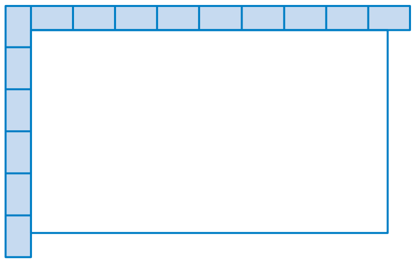
4 A farmer wants to make a rectangular paddock with an area of 4000 m^2. To minimise fencing costs she wants the paddock to have a minimum perimeter.

a Show that the perimeter is given by the equation $P = 2x + \dfrac{8000}{x}$.

b Find the dimensions of the rectangle that will give the minimum perimeter, correct to 1 decimal place.

c Calculate the cost of fencing the paddock, at $48.75 per metre.

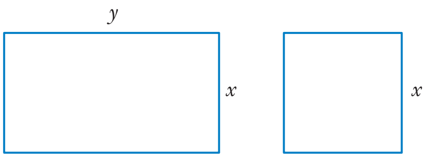
5 Bill wants to put a small rectangular vegetable garden in his backyard using 2 existing walls as part of its border. He has 8 m of garden edging for the border on the other 2 sides. Find the dimensions of the garden bed that will give the greatest area.



6 Find 2 numbers whose sum is 28 and whose product is a maximum.

7 The difference of 2 numbers is 5. Find these numbers if their product is to be minimum.

8 A piece of wire 10 m long is broken into 2 parts, which are bent into the shape of a rectangle and a square as shown. Find the dimensions x and y that make the total area a maximum.



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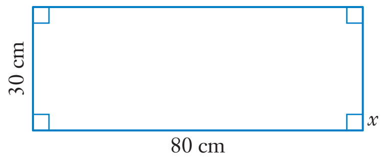
9 A box is made from an 80 cm by 30 cm rectangle of cardboard by cutting out 4 equal squares of side x cm from each corner. The edges are turned up to make an open box.



a Show that the volume of the box is given by the equation $V = 4x^3 - 220x^2 + 2400x$.

b Find the value of x that gives the box its greatest volume.

c Find the maximum volume of the box.

10 The formula for the surface area of a cylinder is given by $S = 2\pi r(r + h)$ where r is the radius of its base and h is its height.

a Show that if the cylinder holds a volume of 54π m^3, the surface area is given by the equation $S = 2\pi r^2 + \dfrac{108\pi}{r}$.

b Hence find the radius that gives the minimum surface area.

11 A silo in the shape of a cylinder is required to hold 8600 m^3 of wheat.

a Find an equation for the surface area of the silo in terms of the base radius.

b Find the minimum surface area required to hold this amount of wheat, to the nearest square metre.

Photo courtesy Margaret Grove

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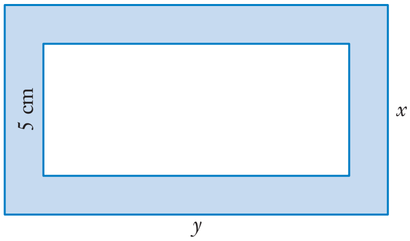
12 A rectangle is cut from a circular disc of radius 6 cm.

a Show that the formula for the area of the rectangle is $A = x\sqrt{144 - x^2}$.

b Find the area of the largest rectangle that can be produced.

13 A poster consists of a photograph bordered by a 5 cm margin. The area of the poster is to be 400 cm^2.

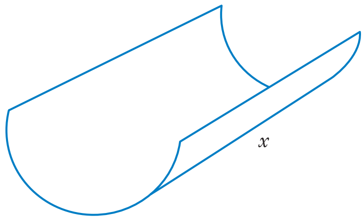


a Show that the area of the photograph is given by the equation $A = 500 - 10x - \dfrac{4000}{x}$.

b Find the maximum area possible for the photograph.

14 A surfboard is in the shape of a rectangle and semicircle, as shown. The perimeter is to be 4 m. Find the maximum area of the surfboard, correct to 2 decimal places.

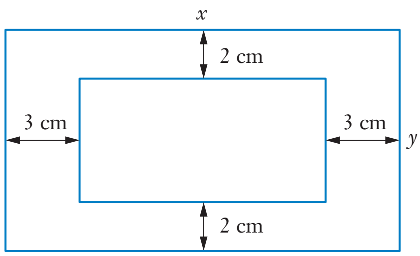
15 A half-pipe is to be made from a rectangular piece of metal of length x m. The perimeter of the rectangle is 30 m.



a Find the dimensions of the rectangle that will give the maximum surface area.

b Find the height from the ground up to the top of the half-pipe with this maximum area, correct to 1 decimal place.

16 The picture frame shown has a border of 2 cm at the top and bottom and 3 cm at the sides. If the total area of the border is to be 100 cm^2, find the maximum area of the frame.



17 A 3 m piece of wire is cut into 2 pieces and bent around to form a square and a circle. Find the size of the 2 lengths, correct to 2 decimal places, that will make the total area of the square and circle a minimum.

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18 Two cars are travelling along roads that intersect at right angles to one another. One starts 200 km away and travels towards the intersection at 80 km h^{-1}, while the other starts at 120 km away and travels towards the intersection at 60 km h^{-1}.

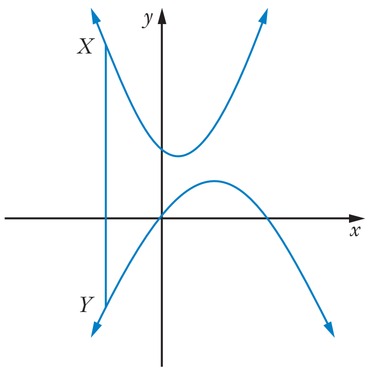
a Show that their distance apart after t hours is given by $d^2 = 10\,000t^2 - 46\,400t + 54\,400$.

b Hence find their minimum distance apart.

19 X is a point on the curve $y = x^2 - 2x + 5$. Point Y lies directly below X and is on the curve $y = 4x - x^2$.

a Show that the distance, d, between X and Y is $d = 2x^2 - 6x + 5$.

b Find the minimum distance between X and Y.



20 A truck travels 1500 km at an hourly cost given by $s^2 + 9000$ cents where s is the average speed of the truck.

a Show that the cost for the trip is given by $C = 1500\left(s + \frac{9000}{s}\right)$.

b Find, to the nearest km h^{-1}, the speed that minimises the cost of the trip.

c Find the cost of the trip to the nearest dollar.

Shutterstock.com/Cesar Witkowski

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CLASS CHALLENGE

HERON'S PROBLEM

One boundary of a farm is a straight river bank, and on the farm stands a house. Some distance away there is a shed. Each is sited away from the river bank. Each morning the farmer takes a bucket from his house to the river, fills it with water, and carries the water to the shed.

Find the position on the river bank that will allow him to walk the shortest distance from house to river to shed. Further, describe how the farmer could solve the problem on the ground with the aid of a few stakes for sighting.

LEWIS CARROLL'S PROBLEM

After a battle at least 95% of the combatants had lost a tooth, at least 90% had lost an eye, at least 80% had lost an arm, and at least 75% had lost a leg. At least how many had lost all four?

Dreamstime.com/Olga Rotula

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6. TEST YOURSELF

For Questions **1**–**4** choose the correct answer **A**, **B**, **C** or **D**.

Practice quiz

Derivatives find-a-word

Second derivative find-a-word

1 A maximum turning point has:

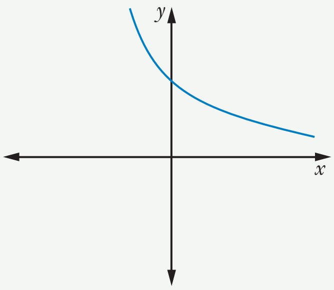
A $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$ **B** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **D** $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$

2 For the graph shown:

A $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} > 0$ **B** $\frac{dy}{dx} > 0$ and $\frac{d^2y}{dx^2} < 0$

C $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} > 0$ **D** $\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$



3 For a horizontal point of inflection:

A $f''(x) = 0$ **B** $f'(x) = 0$ and $f''(x) = 0$

C $f''(x) = 0$ and concavity changes **D** $f'(x) = 0, f''(x) = 0$ and concavity changes

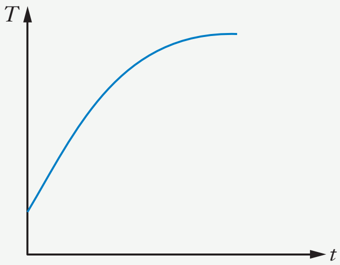
4 The graph below shows temperature T at time t. Which statement describes the shape of the graph?

A The temperature is increasing and the rate of change in temperature is increasing.

B The temperature is decreasing and the rate of change in temperature is increasing.

C The temperature is increasing and the rate of change in temperature is decreasing.

D The temperature is decreasing and the rate of change in temperature is decreasing.



5 Find the stationary points on the curve $y = x^3 + 6x^2 + 9x - 11$ and determine their nature.

6 Find all x values for which the curve $y = 2x^3 - 7x^2 - 3x + 1$ is concave upwards.

7 The height in metres of an object thrown up into the air is given by $h = 20t - 2t^2$, where t is time in seconds. Find the maximum height that the object reaches.

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8 Find the domain over which the curve $y = 5 - 6x - 3x^2$ is decreasing.

9 Find the point of inflection on the curve $y = 2x^3 - 3x^2 + 3x - 2$.

10 A soft drink manufacturer wants to minimise the amount of aluminium in its cans while still holding 375 mL of soft drink. Given that 375 mL has a volume of 375 cm^3:

a show that the surface area of a can is given by $S = 2\pi r^2 + \dfrac{750}{r}$

b find the radius of the can that gives the minimum surface area.

11 For the function $y = 3x^4 + 8x^3 + 6x^2$:

a find any stationary points

b determine their nature

c sketch the curve for the domain $[-3, 3]$

d find the maximum and minimum values of the function in this domain.

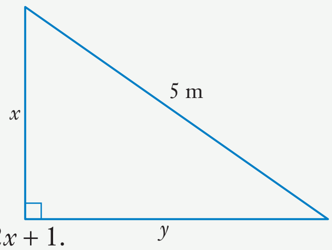
12 A rectangular prism with a square base is to have a surface area of 250 cm^2.

a Show that the volume is given by $V = \dfrac{125x - x^3}{2}$.

b Find the dimensions that will give the maximum volume.

13 The cost to a business of manufacturing x products a week is given by $C = x^2 - 300x + 9000$. Find the number of products that will give the minimum cost each week.

14 A 5 m length of timber is used to border a triangular garden bed, with the other sides of the garden against the house walls.



a Show that the area of the garden is $A = \dfrac{1}{2}x\sqrt{25 - x^2}$.

b Find the greatest possible area of the garden bed.

15 Find any points of inflection on the curve $f(x) = x^4 - 6x^3 + 2x + 1$.

16 Find the maximum value of the curve $y = x^3 + 3x^2 - 24x - 1$ in the domain $[-5, 6]$.

17 A function has $f'(2) < 0$ and $f''(2) < 0$. Sketch the shape of the function near $x = 2$.

18 Sketch the graph of the function $f(x) = xe^{2x}$ showing all features.

19 Sketch the graph of the function $y = 2 \cos 4x$ in the domain $[0, \pi]$.

20 EXT1 Given $f(x) = x^2 - 1$ and $g(x) = x + 1$, sketch the graph of:

a $y = f(x) + g(x)$

b $y = f(x)g(x)$

c $y = \dfrac{g(x)}{f(x)}$

d $y^2 = f(x)g(x)$

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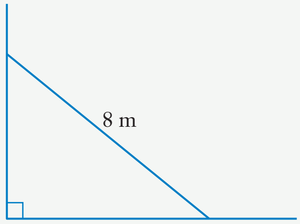
6. CHALLENGE EXERCISE

1 Sketch the curve $y = x(x - 2)^3$ showing any stationary points and points of inflection.

2 EXT1 Find all values of x for which the curve $y = 4x^3 - 21x^2 - 24x + 5$ is increasing.

3 Find the maximum possible area if an 8 m length of fencing is placed across a corner to enclose a triangular space.



4 Find the greatest and least values of $f(x) = 4x^3 - 3x^2 - 18x$ in the domain $[-2, 3]$.

5 Show that the function $f(x) = 2(5x - 3)^3$ has a horizontal point of inflection at $(0.6, 0)$.

6 Two circles have radii r and s such that $r + s = 25$.
Show that the sum of areas of the circles is least when $r = s$.

7 Find the equation of a curve that is always concave upwards with a stationary point at $(-1, 2)$ and y-intercept 3.

EXT1 **8** **a** Given $f(x) = x^2 + 2$, find the stationary point on the reciprocal curve $y = \dfrac{1}{f(x)}$ and determine its nature.

b Find the domain and range of the curve.

c Find the limit of the curve as x approaches $\pm\infty$ and sketch the curve.

9 **a** Show that $y = x^n$ has a stationary point at $(0, 0)$ where n is a positive integer.

b If n is even, show that $(0, 0)$ is a minimum turning point.

c If n is odd, show that $(0, 0)$ is a point of inflection.

10 EXT1 Sketch the graph of $y = \dfrac{f(x)}{g(x)}$ showing any stationary points, inflections and other important features, given $f(x) = x^3$ and $g(x) = x^3 + 1$.

11 Find the minimum and maximum values of $y = \dfrac{x+3}{x^2 - 9}$ in the domain $[-2, 2]$.

12 The cost of running a car at an average speed of V km h^{-1} is given by $c = 100 + \dfrac{V^2}{75}$ cents per hour. Find the average speed (to the nearest km h^{-1}) at which the cost of a 1000 km trip is a minimum.

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