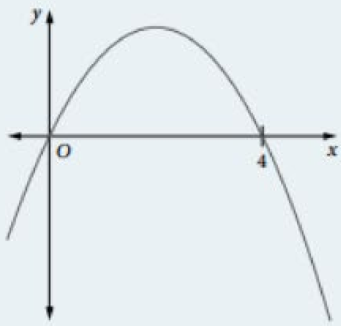
CHAPTER 14
The first and second derivative

14.1 THE SIGN OF THE DERIVATIVE

Remember that the first derivative of a function (as studied earlier, see Chapters 7 and 12) is also known as the gradient function, because it allows you to calculate the gradient at any point on the curve. You have used the process of differentiation to find this gradient function or derivative of a function. This can be positive, negative, zero or undefined. Each possible value has a geometrical application.

Example 1

A sketch of the function $y = 4x - x^2$ is given in the diagram.



(a) Find $\frac{dy}{dx}$ as a function of x.

(b) Complete the following table of values for $\frac{dy}{dx}$:

x	0	1	2	3	4	5
$\frac{dy}{dx}$						

(c) Draw the graph of $\frac{dy}{dx}$ on the same diagram as a graph of y.

(d) For what values of x is **(i)** $\frac{dy}{dx} > 0$ **(ii)** $\frac{dy}{dx} = 0$ **(iii)** $\frac{dy}{dx} < 0$?

(e) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} > 0$.

(f) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} < 0$.

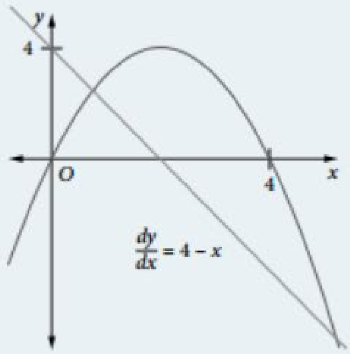
(g) Describe the function $y = 4x - x^2$ where $\frac{dy}{dx} = 0$.

Solution

(a) $\frac{dy}{dx} = 4 - 2x$

(b)

x	0	1	2	3	4	5
$\frac{dy}{dx}$	4	2	0	−2	−4	−6

(c) 

(d) **(i)** $x < 2$ **(ii)** $x = 2$ **(iii)** $x > 2$

(e) Where $\frac{dy}{dx} > 0$, y increases as x increases. The curve slopes up.

(f) Where $\frac{dy}{dx} < 0$, y decreases as x increases. The curve slopes down.

(g) Where $\frac{dy}{dx} = 0$, y neither increases nor decreases. It is at its highest point and the tangent at this point is horizontal.

The sign of the first derivative

- If $\frac{dy}{dx} > 0$ as x increases, the function is an **increasing** function.
- If $\frac{dy}{dx} < 0$ as x increases, the function is a **decreasing** function.
- If $\frac{dy}{dx} = 0$ at a given value of x, the function is **stationary** at that point; the point is called a **stationary point**. At this point, the tangent to the curve is parallel to the x-axis.

Each definition and its converse can be used to determine a function's characteristics:

- If $\frac{dy}{dx} > 0$, then the function is increasing; if the function is increasing, then $\frac{dy}{dx} > 0$.
- If $\frac{dy}{dx} < 0$, then the function is decreasing; if the function is decreasing, then $\frac{dy}{dx} < 0$.
- If $\frac{dy}{dx} = 0$ at a point, then it is a stationary point; at a stationary point, $\frac{dy}{dx} = 0$.

Example 2

For what values of x is the function $f(x) = 2x^3 - 9x^2 - 24x + 1$:

(a) stationary **(b)** increasing **(c)** decreasing?

Solution

Find $f'(x)$: $f'(x) = 6x^2 - 18x - 24$

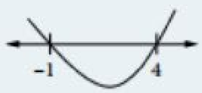
Remove common factor: $f'(x) = 6(x^2 - 3x - 4)$

Factorise: $f'(x) = 6(x+1)(x-4)$

(a) For stationary points, $f'(x) = 0$: $6(x+1)(x-4) = 0$

$x = -1, 4$

Stationary points occur at $x = -1$ or $x = 4$.

(b) Increasing where $f'(x) > 0$: $(x+1)(x-4) > 0$ **OR** Graphically: 

Test $x = 0$: LHS $= 1 \times (-4)$

$= -4$

< 0

Graph is above axis for $x < -1$ or $x > 4$

Function is increasing for $x < -1$ or $x > 4$

Function is not increasing for $-1 < x < 4$

Hence function is increasing for $x < -1$ or $x > 4$.

(c) Decreasing where $f'(x) < 0$, so function is decreasing for $-1 < x < 4$.

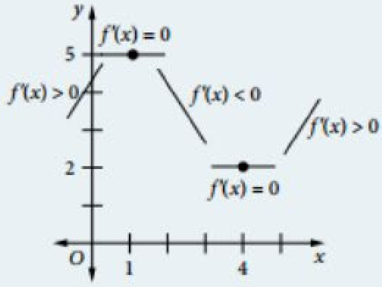
Example 3

Sketch a graph of $f(x)$ such that $f(4)=2$, $f'(4)=0$, $f(1)=5$, $f'(1)=0$, $f'(x)>0$ for all $x<1$ and for all $x>4$; also $f'(x)<0$ for $1<x<4$.

Solution

Mark this information on a number plane:

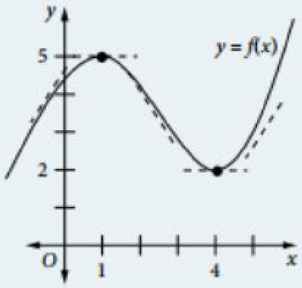
Use straight lines to indicate gradients.



Draw a curve using this information:

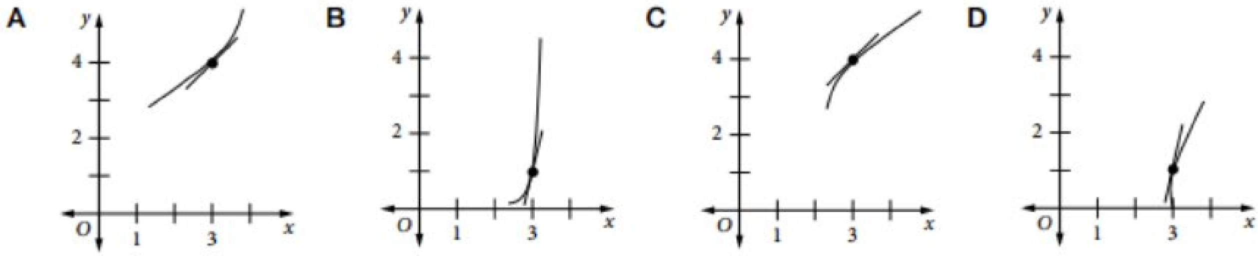
The only points that you know for sure on $y=f(x)$ are $(1,5)$ and $(4,2)$.

From the gradients you know that the curve changes from increasing to decreasing at $(1,5)$ and from decreasing to increasing at $(4,2)$. Fit the curve to this information.



EXERCISE 14.1 THE SIGN OF THE DERIVATIVE

1 A function $f(x)$ has the following properties: $f(3)=4$, $f'(3)=1$. Which sketches fit the graph of $y=f(x)$ near $x=3$?



2 Sketch the graph of $y=f(x)$ with the following properties: $f(1)=0$, $f'(x)=2$ for all x. State the rule that defines the function.

3 Sketch the graph of a function given that $f(2)=0$, $f'(2)=0$, $f'(x)<0$ for all $x<2$, and $f'(x)>0$ for all $x>2$.

4 Sketch the graph of $y=f(x)$, such that $f(3)=3$, $f'(3)=0$, $f(1)=5$, $f'(1)=0$, $f'(x)>0$ for all $x<1$ and for all $x>3$; also $f'(x)<0$ for $1<x<3$.

5 For the function $f(x)=x^2-5x+6$, sketch the graph of $f'(x)$ and hence find the values of x for which:

(a) $f'(x)<0$ **(b)** $f'(x)=0$ **(c)** $f'(x)>0$.

6 Sketch the curve $y=4x-x^2$. For what values of x is $\frac{dy}{dx}=0$? What is the sign of the gradient to the left and right of this point?

7 For the graph of $f(x)=6-3x-x^2$, find the values of x for which the function:

(a) increases when x increases **(b)** decreases when x increases

(c) changes from increasing to decreasing.

8 For the graph of $f(x) = x^3 - 6x^2 + 9x + 2$, find:

(a) $f'(x)$

(b) the values of x for which the function increases when x increases

(c) the values of x for which the function decreases when x increases

(d) the values of x for which the function changes from increasing to decreasing.

9 For the graph of $f(x) = x^3 - 1$, find the values of x for which the function:

(a) increases when x increases **(b)** decreases when x increases

(c) changes from increasing to decreasing.

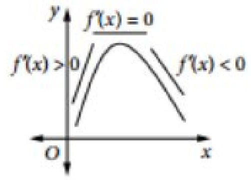
10 For the graph of $f(x) = (x - 1)^2(x + 1)$, find the values of x for which the function is:

(a) stationary **(b)** increasing **(c)** decreasing.

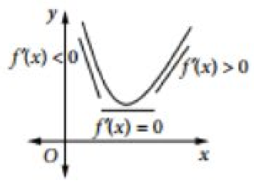
14.2 THE FIRST DERIVATIVE AND TURNING POINTS

In functions you have seen so far, a stationary point is usually the point where the function changes from increasing to decreasing (or from decreasing to increasing), so that the stationary point is the highest (or lowest) point in the neighbourhood. (Here 'neighbourhood' means 'near the point on the graph'.) However, sometimes the function does not change the sign of its gradient at the stationary point. This situation will be considered later.

When a function changes from increasing to decreasing at a stationary point, the sign of $f'(x)$ changes from positive to negative.



When a function changes from decreasing to increasing at a stationary point, the sign of $f'(x)$ changes from negative to positive.



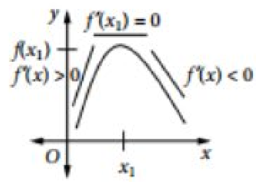
These points are called **turning points**. If the turning point is higher than the other points in its neighbourhood, it is called a **local maximum** turning point. If the point is lower than the other points in its neighbourhood, it is called a **local minimum** turning point.

A turning point of $f(x)$ is a point where the curve $y = f(x)$ is locally a maximum or a minimum.
For a differentiable function $f(x)$, all turning points are stationary points.
(However, note that not all stationary points are turning points.)

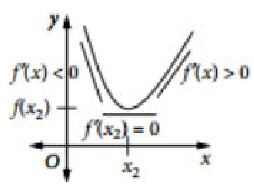
First derivative test for local maxima and minima

A turning point of the differentiable function $f(x)$ may occur when $f'(x) = 0$. The type of turning point will depend on the change in sign of $f'(x)$ as x passes through the abscissa (x-coordinate) of the stationary point, so you need to find the sign of $f'(x)$ on either side of the stationary point.

If $f'(x) = 0$ when $x = x_1$, $f'(x) > 0$ when $x < x_1$, and $f'(x) < 0$ when $x > x_1$, then the sign of $f'(x)$ changes from positive to negative as x passes through x_1. The point $(x_1, f(x_1))$ must be a **local maximum turning point**.



If $f'(x) = 0$ when $x = x_2$, $f'(x) < 0$ when $x < x_2$, and $f'(x) > 0$ when $x > x_2$, then the sign of $f'(x)$ changes from negative to positive as x passes through x_2. The point $(x_2, f(x_2))$ is a **local minimum turning point**.



Thus there are two conditions required to find turning points of the function $y = f(x)$:

1 $f'(x) = 0$ at $x = x_1$, and

2 $f'(x)$ changes sign as x passes through x_1.

The way that $f'(x)$ changes will tell you whether the turning point is a maximum or minimum turning point.

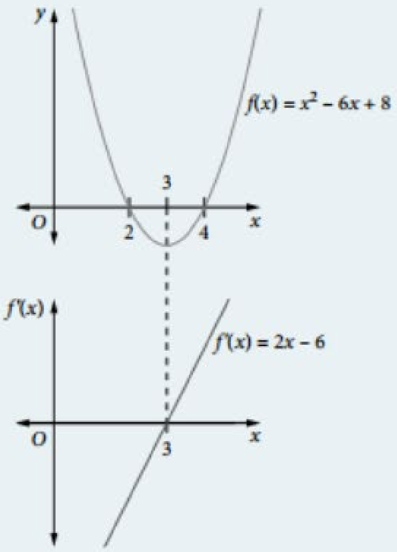
Example 4

The diagrams show the graphs of $f(x) = x^2 - 6x + 8$ and $f'(x) = 2x - 6$ drawn with a common x scale.

(a) Find the coordinates of any stationary points on $f(x)$.

(b) Use the graph of $f'(x)$ to determine the nature of the turning points (i.e. maximum or minimum).

(c) What is the least value of $f(x)$?



Solution

(a) The graph of $f'(x)$ gives that $f'(x) = 0$ when $x = 3$.

$f(3) = 9 - 18 + 8 = -1$

The coordinates of the stationary point are $(3, -1)$.

(b) When $x < 3$, $f'(x) < 0$; when $x > 3$, $f'(x) > 0$.

$f'(x)$ changes from negative to positive as x passes through 3.

$\therefore$ The stationary point is a relative minimum turning point (local minimum).

(c) The least value of $f(x)$ is -1.

Example 5

A function is given by $f(x) = x^3 - 12x + 16$.

(a) Find $f'(x)$.

(b) Find the coordinates of any stationary points.

(c) Determine the nature of the stationary points.

(d) Sketch $y = f(x)$.

Solution

(a) $f'(x) = 3x^2 - 12$

(b) For stationary points, $f'(x) = 0$, so:

$$3(x^2 - 4) = 0$$
$$(x + 2)(x - 2) = 0$$
$$x = -2, 2$$

$f(-2) = -8 + 24 + 16 = 32$ $\quad f(2) = 8 - 24 + 16 = 0$

Stationary points are $(-2, 32)$ and $(2, 0)$.

(c) Consider the stationary point $(-2, 32)$.

At $x = -3$, $f'(-3) = 27 - 12 = 15 > 0$

At $x = -1$, $f'(-1) = 3 - 12 = -9 < 0$

$f'(x)$ changes from +ve to −ve, so $f(x)$ has a maximum turning point at $(-2, 32)$.

Consider the stationary point $(2, 0)$.

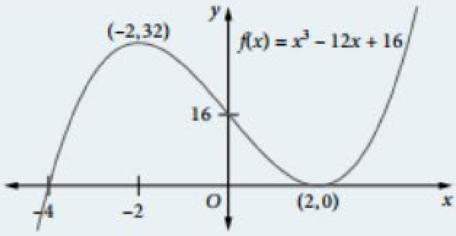
At $x = 1$, $f'(1) = 3 - 12 = -9 < 0$

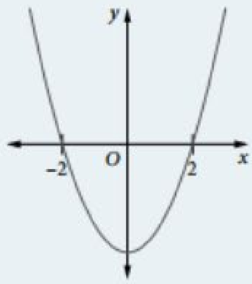
At $x = 3$, $f'(3) = 27 - 12 = 15 > 0$

$f'(x)$ changes from +ve to −ve, so $f(x)$ has a minimum turning point at $(2, 0)$.

$\therefore$ $(-2, 32)$ is a maximum turning point and $(2, 0)$ is a minimum turning point.

(d)





In calculating the change in gradient, a sketch of $f'(x) = 3x^2 - 12$ could have been used to investigate the change in sign of $f'(x)$:

$x < -2$, $f'(x) > 0$
$x > -2$, $f'(x) < 0$: maximum turning point at $(-2, 32)$
$x < 2$, $f'(x) < 0$
$x > 2$, $f'(x) > 0$: minimum turning point at $(2, 0)$

When selecting values of x to substitute into $f'(x)$, you need to pick values near the abscissa (x-coordinate) of the stationary point. But how near is 'near'? In Example **5** you used values that were 1 unit either side of the stationary point, and in this example that was close enough. You could have used -2.1 and -1.9 and reached the same answer, but the calculations would have been more time-consuming.

Calculations for the stationary point at $(-2, 32)$ could have been summarised in tables as follows:

x-value	-3	-2	-1
$f'(x)$	15	0	-9
Sign of $f'(x)$	+	0	−
Direction of curve	↗	→	↘
Shape of curve			

x-value	-2.1	-2	-1.9
$f'(x)$	1.2	0	-1.2
Sign of $f'(x)$	+	0	−
Direction of curve	↗	→	↘
Shape of curve			

This would then allow you to say that the point $(-2, 32)$ is a local maximum.

Turning points—theoretical method

Another method to investigate turning points is to use the vanishingly small positive number ε (the Greek letter epsilon), defined so that ε is infinitesimally small but $\varepsilon > 0$.

To apply the ε method to Example **5**, at the stationary point $(-2, 32)$ consider $x = -2 - \varepsilon$ and $x = -2 + \varepsilon$ as the values of x either side of $x = -2$. Factorise the expression $f'(x) = 3x^2 - 12$ into $f'(x) = 3(x+2)(x-2)$ and examine the values of f':

$$f'(-2-\varepsilon) = 3(-2-\varepsilon+2)(-2-\varepsilon-2) = 3(-\varepsilon)(-4-\varepsilon) > 0, \text{ because } (-)\times(-) = (+)$$

$$f'(-2+\varepsilon) = 3(-2+\varepsilon+2)(-2+\varepsilon-2) = 3(\varepsilon)(-4+\varepsilon) < 0, \text{ because } (+)\times(-) = (-)$$

$f'(x)$ changes from +ve to −ve, so $f(x)$ has a maximum turning point at $(-2, 32)$.

Example 6

The function $y = 3x^2 - x^3$ is defined on the domain $-2 \le x \le 4$.

(a) Find the stationary points and determine their nature.
(b) Find the greatest and least values of y in the domain.
(c) Sketch the function for the given domain.

Solution

(a) $y = 3x^2 - x^3$

$\frac{dy}{dx} = 6x - 3x^2$

$= 3x(2 - x)$

For stationary points, $\frac{dy}{dx} = 0$

$3x(2 - x) = 0$

$\therefore x = 0 \quad \text{or} \quad x = 2$

Stationary points are $(0, 0)$ and $(2, 4)$.

For $\varepsilon > 0$:

$x = 0 - \varepsilon, \quad \frac{dy}{dx} = 3(-\varepsilon)(2 - (-\varepsilon)) = -3\varepsilon(2 + \varepsilon) < 0$

$x = 0 + \varepsilon, \quad \frac{dy}{dx} = 3\varepsilon(2 - \varepsilon) > 0$

$\frac{dy}{dx}$ changes from –ve to +ve, so $(0, 0)$ is a minimum turning point.

$x = 2 - \varepsilon, \quad \frac{dy}{dx} = 3(2 - \varepsilon)(2 - (2 - \varepsilon)) = 3(2 - \varepsilon)(\varepsilon) > 0$

$x = 2 + \varepsilon, \quad \frac{dy}{dx} = 3(2 + \varepsilon)(2 - (2 + \varepsilon)) = 3(2 + \varepsilon)(-\varepsilon) < 0$

$\frac{dy}{dx}$ changes from +ve to –ve, so $(2, 4)$ is a maximum turning point.

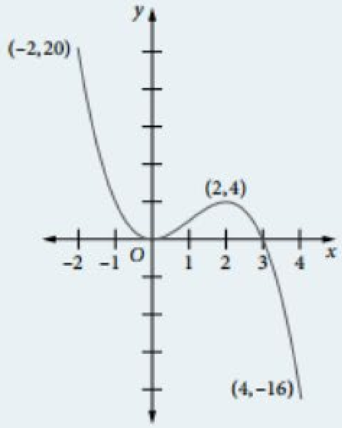
(b) The turning points give the local maximum and minimum values. The values at the endpoints of a given domain may be greater than or less than these local values. In this example you need to find the value of y at $x = -2$ and $x = 4$.

$x = -2: \quad y = 12 + 8 = 20$

$x = 4: \quad y = 48 - 64 = -16$

Hence the greatest value of the function in the domain is 20 and the least value is –16. These occur at the endpoints of the domain.

(c)



MAKING CONNECTIONS

Tangent to a curve at turning points

Explore the changing gradient of the tangent to the curve near its turning points.

EXERCISE 14.2 THE FIRST DERIVATIVE AND TURNING POINTS

1 A function is given by $f(x) = x^2 - 6x + 8$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$.

2 If $f'(x) = x^2 - 5x - 6$ then stationary points may occur when:

A $x = 1, -6$ **B** $x = -2, -3$ **C** $x = -1, 6$ **D** $x = -3, 2$

3 A function is given by $f(x) = x^3 - 6x^2 + 16$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$.

4 A function is given by $f(x) = x^3 - 9x^2 + 15x + 16$. You are asked to find the stationary points and determine their nature. Four steps in answering this question are shown below. Indicate whether each step is correct or incorrect.

(a) $f'(x) = 3x^2 - 18x + 15$ **(b)** Stationary points occur when $x = -1, 5$.
(c) $(1, 23)$ is a maximum turning point. **(d)** $(5, -9)$ is a minimum turning point.

5 A function is given by $f(x) = 2x^3 - 15x^2 + 36x$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$.

6 A function is given by $f(x) = x^3 - x^2 - x + 1$.

(a) Find $f'(x)$. **(b)** Find the coordinates of any stationary points and determine their nature.
(c) Sketch $y = f(x)$ over the domain $-2 \le x \le 2$, showing the turning points.

7 Find the maximum value of $5x - 2x^2$.

8 Find the minimum value of $x(x - 2) + 3$.

9 Sketch the curve $y = x^3 - 6x^2$ over the domain $-1 \le x \le 6$, showing the maximum and minimum turning points.

10 Find the turning points of $y = 2x^3 + 3x^2 - 12x + 7$. Hence sketch the curve, showing the turning points and the y-intercept.

11 Sketch the curve $y = (2 - x)(1 + x^2)$, locating the turning points and the points where it crosses the coordinate axes over the domain $-1 \le x \le 3$.

12 Consider the function $f(x) = 9x(x - 2)^2, -1 \le x \le 3$. Find the values of x for which:

(a) $f'(x) = 0$ **(b)** $f'(x) > 0$ **(c)** $f'(x) < 0$
(d) Sketch the graph of $f(x)$ and state its range and greatest and least values.

13 Prove that the parabola $y = ax^2 + bx + c$ has a turning point at $x = \frac{-b}{2a}$.

14 Show that the hyperbola $y = \frac{1}{x}$ has no turning points. Also show that its gradient is always negative throughout its domain.

14.3 THE SECOND DERIVATIVE AND CONCAVITY

Differentiating a function once to obtain $f'(x)$ or $\frac{dy}{dx}$ gives you the first derivative of the original function. Differentiating the first derivative gives you $f''(x)$ or $\frac{d^2y}{dx^2}$, which is called the **second derivative** of the original function. The second derivative is the rate of change of the first derivative (i.e. of the gradient function): $\frac{d}{dx}\big(f'(x)\big)$. Differentiating again will give the third derivative, and so on. The differentiation process may be continued for as long as a derivative exists.

Notation

Several different notations can be used for derivatives. If $y = f(x)$, then:

- the first derivative can be written $\frac{dy}{dx}$, $f'(x)$, $\frac{d}{dx}\big(f(x)\big)$ or y'
- the second derivative can be written $\frac{d}{dx}\left(\frac{dy}{dx}\right)$, $\frac{d^2y}{dx^2}$, $f''(x)$, $\frac{d}{dx}\big(f'(x)\big)$ or y''.

Example 7

Find the second derivative of each function.

(a) $y = 4x^3 - 2x^2 + 3x + 7$ **(b)** $f(x) = (2x + 1)^5$ **(c)** $y = \frac{x^2}{x+1}$

Solution

(a) $y = 4x^3 - 2x^2 + 3x + 7$:

$$\frac{dy}{dx} = 12x^2 - 4x + 3$$

$$\frac{d^2y}{dx^2} = 24x - 4$$

(b) $f(x) = (2x+1)^5$:

$$f'(x) = 5(2x+1)^4 \times 2$$
$$= 10(2x+1)^4$$
$$f''(x) = 10 \times 4(2x+1)^3 \times 2$$
$$= 80(2x+1)^3$$

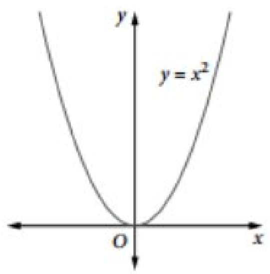
(c) $y = \frac{x^2}{x+1}$:

$$y' = \frac{2x(x+1) - x^2 \times 1}{(x+1)^2}$$
$$= \frac{2x^2 + 2x - x^2}{(x+1)^2}$$
$$= \frac{x^2 + 2x}{(x+1)^2}$$

$$y'' = \frac{(2x+2)(x+1)^2 - \left(x^2+2x\right) \times 2(x+1)}{(x+1)^4}$$
$$= \frac{2(x+1)^3 - 2(x+1)\left(x^2+2x\right)}{(x+1)^4}$$
$$= \frac{2(x+1)\left((x+1)^2 - \left(x^2+2x\right)\right)}{(x+1)^4}$$
$$= \frac{2\left(x^2 + 2x + 1 - x^2 - 2x\right)}{(x+1)^3}$$
$$= \frac{2}{(x+1)^3}$$

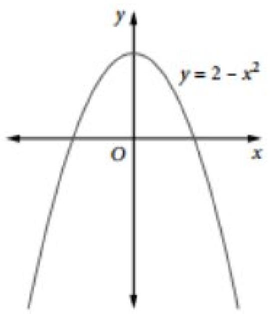
Concavity

The **concavity** of a function describes the general curvature of a graph of a non-linear function. Graphs can be 'concave upwards' and 'concave downwards':



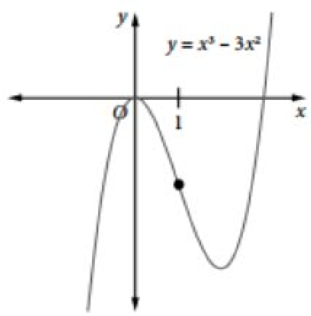
The function is concave up.

Note: This has a minimum turning point.



The function is concave down.

Note: This has a maximum turning point.



The left-hand part is concave down.

The right-hand part is concave up.

Concavity changes at $x = 1$.

> The second derivative is the rate at which the first derivative is changing. This gives information about the concavity of a function.

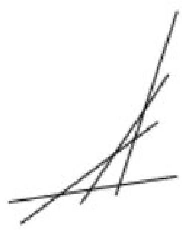
In each diagram below, a series of tangents have been drawn.

1 The tangents have a positive gradient and the gradient is increasing from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} > 0$.

The rate at which the gradient is increasing $= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$.

Because the gradient is increasing, $\frac{d^2y}{dx^2} > 0$: the curve is concave up.



2 The tangents have a negative gradient and the gradient is increasing (becoming less negative) from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} < 0$.

Because the gradient is increasing, $\frac{d^2y}{dx^2} > 0$: the curve is concave up.

3 The tangents have a negative gradient and the gradient is decreasing (becoming more negative) from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} < 0$.

Because the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$: the curve is concave down.

4 The tangents have a positive gradient and the gradient is decreasing from left to right.

If the gradient of the tangent $= \frac{dy}{dx}$, then $\frac{dy}{dx} > 0$.

Because the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$: the curve is concave down.

5 The gradient is positive.

Initially the gradient is decreasing, $\frac{d^2y}{dx^2} < 0$, but then it starts increasing, $\frac{d^2y}{dx^2} > 0$.

This means that at some point $\frac{d^2y}{dx^2} = 0$. This is also where the concavity changes from concave down to concave up, so this point is called a **point of inflection**.

6 The gradient is negative.

Initially the gradient is increasing, $\frac{d^2y}{dx^2} > 0$, but then it starts decreasing, $\frac{d^2y}{dx^2} < 0$.

This means that at some point $\frac{d^2y}{dx^2} = 0$. This is also where the concavity changes from concave down to concave up, so this is a point of inflection.

The sign of the second derivative

- If $\frac{d^2y}{dx^2} > 0$ on an interval then the curve is concave upwards on that interval.
- If $\frac{d^2y}{dx^2} < 0$ on an interval then the curve is concave downwards on that interval.
- If $\frac{d^2y}{dx^2} = 0$ at a point on the curve and the concavity changes at this point, then the point is called a **point of inflection**.

MAKING CONNECTIONS

The second derivative and points of inflection

Use graphing software to explore the changing value of the second derivative of a function.

Example 8

For what values of x is the curve given by $y = x^3 - 3x^2 + 6x + 3$:

(a) concave up **(b)** concave down?

(c) Find the coordinates of the point of inflection. **(d)** Sketch the curve.

Solution

Find $\frac{dy}{dx}$: $\frac{dy}{dx} = 3x^2 - 6x + 6$

Find $\frac{d^2y}{dx^2}$: $\frac{d^2y}{dx^2} = 6x - 6$

(a) Concave up, $\frac{d^2y}{dx^2} > 0$: $6x - 6 > 0$
$x > 1$

The curve is concave up for $x > 1$.

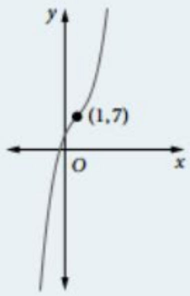
(b) Concave down, $\frac{d^2y}{dx^2} < 0$: $x < 1$

The curve is concave down for $x < 1$.

(c) Inflection point, $\frac{d^2y}{dx^2} = 0$: $x = 1$

$x < 1$, curve is concave down
$x > 1$, curve is concave up
$\therefore$ concavity changes at $x = 1$
$x = 1, y = 1 - 3 + 6 + 3 = 7$
$\therefore$ point of inflection is $(1,7)$

(d)



EXERCISE 14.3 THE SECOND DERIVATIVE AND CONCAVITY

1 Find $f''(x)$ for each function.

(a) $f(x) = 3x^2 + 5x + 6$ **(b)** $f(x) = x^3 + 2x^2 + 4x + 2$ **(c)** $f(x) = 24 - x^2$
(d) $f(x) = x^5 + 2x^3 - 4x$ **(e)** $f(x) = 12 - x^4 + 2x^2$ **(f)** $f(x) = 5x - 4$

2 Given $f(x) = x^6 + 3x^3 - 4x + 2$, $f''(x) = \ldots$

A $6x^5 + 3x^2 - 4$ **B** $6x^5 + 9x^2 - 4$ **C** $30x^4 + 6x$ **D** $30x^4 + 18x$

3 Given $y = \frac{x^2 - 1}{x}$, find $\frac{d^2y}{dx^2}$. Indicate whether each statement below is a correct or incorrect step in finding $\frac{d^2y}{dx^2}$.

(a) $y = x - \frac{1}{x}$ **(b)** $\frac{dy}{dx} = \frac{x^2 - 1}{x^2}$ **(c)** $\frac{dy}{dx} = 1 + \frac{1}{x^2}$ **(d)** $\frac{d^2y}{dx^2} = \frac{-2}{x^2}$

4 Find $\frac{d^2y}{dx^2}$ given: **(a)** $y = \sqrt{x}$ **(b)** $y = \sqrt{x - 2}$ **(c)** $y = x\sqrt{x^2 + 1}$ **(d)** $y = \frac{1}{x}$

(e) $y = \frac{1}{x + 1}$ **(f)** $y = \frac{x}{x + 3}$ **(g)** $y = \frac{x^2 + 1}{\sqrt{x}}$ **(h)** $y = (x^2 - 1)\sqrt{x}$ **(i)** $y = \frac{\sqrt{x - 1}}{x + 1}$

5 For what values of x is $y = 5x^2 - 1$ concave up?

6 For what values of x is $y = 6 - 3x^2$ concave down?

7 For what values of x is the curve $y = x^3 + 6x^2 - x + 4$:

(a) concave up **(b)** concave down? **(c)** Find the coordinates of the point of inflection.

8 For what values of x is the graph of $y = \sqrt{x + 1}$ concave down? Explain why there is no point of inflection.

9 For what values of x is the curve $y = \frac{1}{x}$:

(a) concave up **(b)** concave down? **(c)** Explain why there is no point of inflection.

10 Explain why the graph of $y = \frac{1}{x^2}$ is concave up over its domain.

14.4 THE SECOND DERIVATIVE AND TURNING POINTS

Just as you compared the graphs of $f(x)$ and $f'(x)$ drawn together on the same horizontal scale, you now need to consider the graphs of $f(x)$, $f'(x)$ and $f''(x)$ together.

Consider: $y = f(x) = x^3 - x^2 - x + 1$

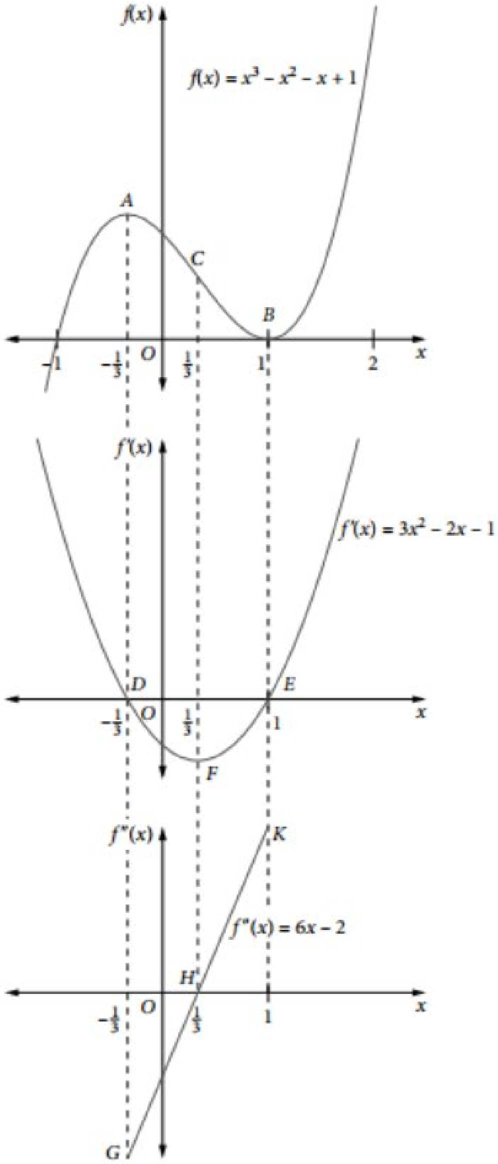
Differentiate: $\frac{dy}{dx} = y' = f'(x) = 3x^2 - 2x - 1$

Differentiate again: $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = y'' = f''(x) = 6x - 2$

For stationary points, $f'(x) = 0$: $3x^2 - 2x - 1 = 0$

Factorise: $(3x+1)(x-1) = 0$

Solve: $x = -\frac{1}{3}, 1$



On the graph, the vertical lines ADG and BEK correspond to the lines $x = -\frac{1}{3}$ and $x = 1$ respectively.

Using the first derivative test (or the graph of $f'(x)$), you can say that A is a maximum turning point and that B is a minimum turning point.

- Consider where the line ADG cuts the three curves. The point A on $f(x)$ is a maximum turning point. The point D on $f'(x)$ is where $f'(x)$ cuts the x-axis, that is, $f'(x) = 0$. At the point G on $f''(x)$, $f''(x) < 0$. Hence the curve $y = f(x)$ is concave down at a maximum turning point.
- Consider where the line BEK cuts the three curves: at point B, $f(x)$ is a minimum turning point; at point E, $f'(x)$ cuts the x-axis, i.e. $f'(x) = 0$; at point K, $f''(x) > 0$. Hence the curve $y = f(x)$ is concave up at a minimum turning point.
- Consider where the line CFH cuts the three curves: at point C, $f(x)$ seems to have its steepest tangent; at point F, $f'(x)$ has its least value, i.e. the gradient of $f(x)$ is at its most negative (steepest); at point H, $f''(x) = 0$. The concavity changes either side of C, so C is a point of inflection on $y = f(x)$.

For $y = f(x) = x^3 - x^2 - x + 1$, you can say that the function has a maximum turning point at $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$, a minimum turning point at $(1, 0)$, is concave down for $x < \frac{1}{3}$, is concave up for $x > \frac{1}{3}$ and has a point of inflection at $\left(\frac{1}{3}, \frac{16}{27}\right)$.

Second derivative test for turning points

- If $\frac{dy}{dx} = 0$ at (x_1, y_1) and $\frac{d^2y}{dx^2} < 0$ then the point (x_1, y_1) is a maximum turning point.
- If $\frac{dy}{dx} = 0$ at (x_2, y_2) and $\frac{d^2y}{dx^2} > 0$ then the point (x_2, y_2) is a minimum turning point.
- If $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at (x_3, y_3) then the point may be a turning point **OR** it may be a horizontal point of inflection. Further investigation is needed.

Point of inflection test

- If $\frac{d^2y}{dx^2}=0$ at (x_4, y_4) and the concavity changes either side of this point, then (x_4, y_4) is a point of inflection.
- If $\frac{dy}{dx}=0$ at a point of inflection then it is called a horizontal point of inflection.

Example 9

Investigate the stationary points of:

(a) $y=x^4$ **(b)** $y=3x^2-3x-x^3$

Solution

(a) $y=x^4$: $\frac{dy}{dx}=4x^3$; $\frac{d^2y}{dx^2}=12x^2$

For stationary points, $\frac{dy}{dx}=0$: $x=0$

At $x=0$, $\frac{d^2y}{dx^2}=0$, so you can't identify the nature of this stationary point.

Check the sign of the first derivative: For $x<0$, e.g. $x=-1$: $\frac{dy}{dx}=-4<0$

For $x>0$, e.g. $x=1$: $\frac{dy}{dx}=4>0$

The gradient changes from negative to positive through $x=0$, so $(0,0)$ is a minimum turning point.

Alternative method:

Check the sign of the second derivative: For $x<0$, e.g. $x=-1$: $\frac{d^2y}{dx^2}=12>0$

For $x>0$, e.g. $x=1$: $\frac{d^2y}{dx^2}=12>0$

The curve is concave up on both sides of the stationary point, so $(0,0)$ is a minimum turning point.

(b) $y=3x^2-3x-x^3$:
$$\begin{aligned}\frac{dy}{dx}&=6x-3-3x^2\\&=-3\left(x^2-2x+1\right)\\&=-3(x-1)^2\end{aligned}\qquad \frac{d^2y}{dx^2}=6-6x$$

For stationary points, $\frac{dy}{dx}=0$: $(x-1)^2=0$, hence $x=1$

At $x=1$, $\frac{d^2y}{dx^2}=0$, so you can't identify the nature of this stationary point.

Check the sign of the second derivative: For $x<1$, e.g. $x=0.1$: $\frac{d^2y}{dx^2}=6-0.6$
$=5.4>0$

For $x>1$, e.g. $x=1.1$: $\frac{d^2y}{dx^2}=6-6.6$
$=-0.6<0$

The concavity changes either side of $x=1$, so $(1,-1)$ is a horizontal point of inflection.

Example 10

Consider the graph of $y = 2x^4 - x + 1$.

(a) Find any turning points and points of inflection. **(b)** For what values of x is the curve concave up?

(c) On the same set of axes sketch y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

Solution

(a) $y = 2x^4 - x + 1$: $\quad \frac{dy}{dx} = 8x^3 - 1$

For stationary points, $\frac{dy}{dx} = 0$: $\quad 8x^3 - 1 = 0$, hence $x^3 = \frac{1}{8}$ and so $x = \frac{1}{2}$.

At $x = \frac{1}{2}$, $y = 2 \times \frac{1}{2^4} - \frac{1}{2} + 1$

$= 1\frac{3}{8}$ $\quad \therefore$ stationary point is at $\left(\frac{1}{2}, 1\frac{3}{8}\right)$.

Find $\frac{d^2y}{dx^2}$: $\quad \frac{d^2y}{dx^2} = 24x^2$

At $x = \frac{1}{2}$, $\frac{d^2y}{dx^2} = 6 > 0 \quad \therefore \left(\frac{1}{2}, 1\frac{3}{8}\right)$ is a minimum turning point.

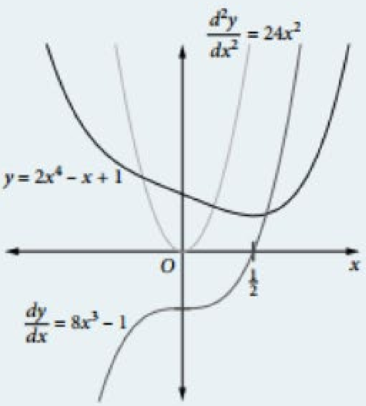
$\frac{d^2y}{dx^2} = 0$ at $x = 0$, so check concavity to see if there is an inflection point at $(0, 1)$:

$x < 0$: $\frac{d^2y}{dx^2} > 0 \qquad x > 0$: $\frac{d^2y}{dx^2} > 0$

The concavity does not change either side of $x = 0$, so the curve does not have a point of inflection at $(0, 1)$.

(b) Concave up when $\frac{d^2y}{dx^2} > 0$, hence concave up for all real x.

(c)



This example shows that $\frac{d^2y}{dx^2} = 0$ is not enough to confirm a point of inflection. You must also check that the concavity changes at the point.

Example 11

Sketch the graph of $y = x^3(4 - x)$, showing any turning points and points of inflection. For what values of x is the curve concave down?

Solution

$y = 4x^3 - x^4$: $\quad \frac{dy}{dx} = 12x^2 - 4x^3$ $\quad \frac{d^2y}{dx^2} = 24x - 12x^2$

$\qquad = 4x^2(3 - x)$ $\qquad = 12x(2 - x)$

For stationary points, $\frac{dy}{dx} = 0$: $\quad 4x^2(3 - x) = 0$

$x = 0$ or 3

$y = 0$ or 27 $\quad \therefore$ stationary points at $(0, 0)$ and $(3, 27)$

At $x = 0$, $\frac{d^2y}{dx^2} = 0$. Investigate further: $\quad x = -1$: $\frac{d^2y}{dx^2} = -12(2 + 1) < 0$

$x = 1$: $\frac{d^2y}{dx^2} = 12(2 - 1) > 0$

The sign of $\frac{d^2y}{dx^2}$ changes, so the concavity changes. Hence $(0, 0)$ is a horizontal point of inflection.

At $x = 3$: $\quad \frac{d^2y}{dx^2} = 36(2 - 3) < 0$ $\quad$ Hence $(3, 27)$ is a relative maximum turning point.

For points of inflection, $\frac{d^2y}{dx^2} = 0$: $\quad 12x(2 - x) = 0$

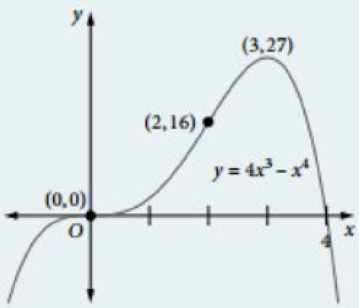
$x = 0$ or 2

$(0, 0)$ has already been identified as a horizontal point of inflection. For $(2, 16)$:

$x = 1$: $\frac{d^2y}{dx^2} = 12(2 - 1) > 0$ $\qquad x = 3$: $\frac{d^2y}{dx^2} = 36(2 - 3) < 0$

The sign of $\frac{d^2y}{dx^2}$ changes, so the concavity changes. Hence $(2, 16)$ is a point of inflection.

Curve is concave downwards for $x < 0$ and $x > 2$.



Summary of tests for turning points and points of inflection

Stationary point: $\frac{dy}{dx} = 0$

Using the first derivative only:

Turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign on passing through the stationary point.

Maximum turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign from positive to negative on passing through the stationary point.

Minimum turning point: $\frac{dy}{dx} = 0$, $\frac{dy}{dx}$ changes sign from negative to positive on passing through the stationary point.

Using the first and second derivatives:

Turning point: $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2}$ does not change sign on passing through the stationary point.

Maximum turning point: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} < 0$ at the stationary point.

Minimum turning point: $\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} > 0$ at the stationary point.

Point of inflection: $\frac{d^2y}{dx^2} = 0, \frac{d^2y}{dx^2}$ changes sign on passing through the stationary point (i.e. concavity changes).

Special situation:

$\frac{dy}{dx} = 0, \frac{d^2y}{dx^2} = 0$ at a point: may be a turning point **or** a horizontal point of inflection. You must check to see if either the gradient or the concavity changes.

Global maxima and minima

In Example **10** the minimum turning point gives the least value of the function over its domain. It is called the global minimum value of the function. The function has no greatest value.

In Example **11** the maximum turning point gives the greatest value of the function over its domain. It is called the global maximum value of the function. The function has no least value.

The greatest and least values of a function, also called the global maxima and global minima, may occur at the endpoints of the domain or at the turning points of the function.

When asked to find the global maximum or global minimum value of a function, as well as considering the values of the function at the turning points, you also need to find the value of the function at the endpoints of the given domain (or the natural domain if no restrictions are given). You then compare the value of the function at these points to find the global maximum and the global minimum.

Example 12

(a) Find the coordinates of the stationary points of $y = x^2e^x$ and determine their nature.

(b) Find the coordinates of any points of inflection.

(c) Sketch the graph of this function.

(b) Find the global maximum and global minimum values of this function.

Solution

(a) $y = x^2e^x$

Differentiate: $\frac{dy}{dx} = 2xe^x + x^2e^x$

Factorise: $\frac{dy}{dx} = xe^x(2+x)$

For stationary points, $\frac{dy}{dx} = 0$: $x = 0, -2$

$y = 0, 4e^{-2}$

Find second derivative: $\frac{d^2y}{dx^2} = 2e^x + 2xe^x + 2xe^x + x^2e^x$

Factorise: $\frac{d^2y}{dx^2} = e^x\left(2+4x+x^2\right)$

$x = 0$: $\frac{d^2y}{dx^2} = 2 > 0$ so minimum turning point at (0, 0).

$x = -2$: $\frac{d^2y}{dx^2} = -2e^{-2} < 0$ so maximum turning point at $(0, 4e^{-2})$.

(b) Require $\frac{d^2y}{dx^2}=0$: $\quad e^x\left(2+4x+x^2\right)=0$

$e^x>0$: $\quad x^2+4x+2=0$

$$x=\frac{-4\pm\sqrt{8}}{2}$$
$$=-2\pm\sqrt{2}$$
$$\approx -0.586, -3.41$$

$x=-1$: $\quad \frac{d^2y}{dx^2}=-e^{-1}<0$

$x=0$: $\quad \frac{d^2y}{dx^2}=2>0$ and concavity changes at $x=-0.586$

Hence a point of inflection when $x=-0.586$

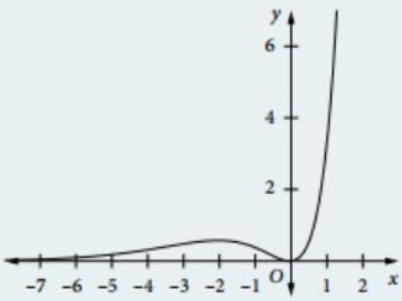
$x=-4$: $\quad \frac{d^2y}{dx^2}=2e^{-4}>0$

$x=-3$: $\quad \frac{d^2y}{dx^2}=-e^{-3}<0$ and concavity changes at $x=-3.41$

Hence a point of inflection when $x=-3.41$

The points of inflection are $(-0.586, 0.191)$ and $(-3.41, 0.384)$.

(c)



(d) From the graph the global minimum value is 0. There is no global maximum value.
Algebraically, as $x\to-\infty$, $y\to 0$ from above (asymptote).
As $x\to\infty$, y increases without bound. Hence the global minimum value occurs at the minimum turning point.

EXERCISE 14.4 THE SECOND DERIVATIVE AND TURNING POINTS

1 For $y=2x^3+3x^2-12x+2$, find any stationary points and determine their nature. Sketch the curve, showing the turning points and any points of inflection.

2 For what values of x is the graph of the function $f(x)=x^3-3x^2+1$ concave up?

A $x>1$ **B** $x>-2$ **C** $x<1$ **D** $x<-2$

3 Find the local maxima, minima and points of inflection of $f(x)=x^2(3-x)$. Sketch the graph of f.

4 Sketch the graph of $8y=8+8x^2-x^4$, showing any turning points. What is the greatest value of the function?

5 A function $f(x)$ is defined by $y=x^3(x-2)$.

(a) Find the coordinates of the turning points of $y=f(x)$.

(b) Find the coordinates of the points of inflection.

(c) Hence sketch the graph of $y=f(x)$, showing the turning point, the points of inflection and the points where the curve meets the x-axis.

(d) What is the minimum value of $f(x)$ for $-1\le x\le 3$?

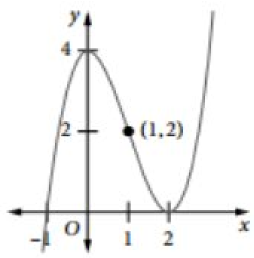
6 If $f(x) = x^3 - 6x^2 + 2$, find the values of x for which: **(a)** $f''(x) = 0$ **(b)** $f''(x) > 0$ **(c)** $f''(x) < 0$

7 Find the coordinates of the points of inflection of $y = x^4 - 2x^3 - 12x^2$. For what values of x is the curve concave up?

8 Let $f(x) = x^4 - x^2$.

(a) Find the coordinates of the points where the curve crosses the axes.
(b) Find the coordinates of the stationary points and determine their nature.
(c) Find the coordinates of the points of inflection.
(d) Sketch the graph of $y = f(x)$ for $-1.5 \le x \le 1.5$, indicating clearly the intercepts, stationary points and points of inflection.
(e) For what values of x is the curve concave down?

9 The graph represents a function of the form $y = ax^3 + bx^2 + cx + d$, where a, b, c and d are real numbers, $a \neq 0$. $(1, 2)$ is a point of inflection on the curve. Indicate whether each statement below is correct or incorrect for this graph.



(a) The curve is concave up for $x > 1$.
(b) $\frac{dy}{dx} = mx(x-2)$ where m is a positive constant. **(c)** The least value of y is zero.
(d) The coordinates of the point of inflection can be found by taking the average of the coordinates of the maximum and minimum turning points.

10 Find the turning points and points of inflection of $y = -x^3 + 3x^2 - 3x$ and sketch its graph. Show that it crosses the x-axis at one point only. Show that $\frac{dy}{dx} < 0$ for all x except $x = 1$.

11 **(a)** Find the stationary points of $y = x^4 - 4x + 3$ and determine their nature.
(b) Show that $y = x^4 - 4x + 3$ has no points of inflection. Comment on the concavity of the curve.
(c) What is the global minimum value of this function?

12 **(a)** Find the greatest and least values of the function $y = x^3 - 6x^2 + 6x$ over the domain $[0, 6]$.
(b) Sketch the graph of the function.

13 The revenue function for a magazine is given by $R = 4500x - 500x^2$, where x is the cost per issue of the magazine. What will be the cost per issue of the magazine to achieve maximum revenue?

14 The revenue equation for a manufacturer is $R = \frac{80x - x^2}{4}$, where x is the number of units sold. How many units must be sold to achieve maximum revenue?

15 A supplier has a monopoly on sales of books. The supplier's profit function is given by $P = 396x - 2.2x^2 - 400$, where x is the number of books sold.

(a) How many books must the supplier sell to maximise the profit?
(b) What is the maximum profit?
(c) If the government imposes a new 'monopoly tax' of \$22 per book on the supplier, what is the new profit equation?
(d) Under the monopoly tax, how many books must the supplier now sell to maximise the profit? What is the new maximum profit?

16 $G = f(t)$ is the Gross Domestic Product (GDP) for Australia, where t is the number of years after 2012. The GDP growth rate $f'(t)$ for three years is given in the table.

Year	2012	2013	2014
$f'(t)$	3.9	3.2	3.1

(a) Is $f'(t)$ a decreasing or increasing function?
(b) What can you say about change in $f(t)$? **(c)** Is $f(t)$ concave up or concave down?
(d) There are two different predictions for the rate of growth in 2015: that the growth rate will be 3.0, or that the growth rate will be 3.2. Sketch $G = f(t)$ for each growth rate.
(e) For which growth rate does the concavity of $f(t)$ change? What is the name of the kind of point at which this change occurs?
(f) If the growth rate for 2015 is actually 2.9, what can you say about the GDP for Australia?

14.5 PROBLEM SOLVING WITH DERIVATIVES

A function may not always be given algebraically. Sometimes you must interpret the information given to construct the function. Remember to show clearly what each variable represents. It often helps to draw a diagram of the situation being considered.

Example 13

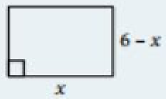
A piece of wire 12 cm long is bent in the shape of a rectangle. Find the maximum area of the rectangle.

Solution

Let $A(x)$ cm^2 be the area of the rectangle. Let the side lengths be x cm and y cm.

Because the wire is 12 cm long: $2x + 2y = 12 \quad \therefore y = (6 - x)$

Because the wire is 12 cm, the longest side of the rectangle cannot be longer than 6 cm.



Area of rectangle: $A(x) = x(6 - x)$ for $0 < x < 6$

$A(x) = 6x - x^2$

Differentiate: $A'(x) = 6 - 2x$

For stationary points, $A'(x) = 0$: $6 - 2x = 0$

$x = 3$

Differentiate again: $A''(x) = -2 < 0$ for all x

Hence $A(x)$ is concave down for all values of x in the domain. It has a maximum value when $x = 3$.

$A(3) = 3 \times 3 = 9\,\text{cm}^2$

The maximum area of the rectangle is 9 cm^2.

Example 14

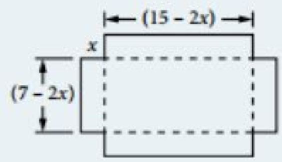
A sheet of cardboard measures 15 cm by 7 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the edge length of the squares that were cut out to give the box a maximum volume.

Solution

Let the edge length of the squares that were cut out be x cm.

The dimensions of the base of the box will be $(15 - 2x)$ cm and $(7 - 2x)$ cm.

The height will be x cm.



Let the volume of the box be $V(x)$ cm^3. $\therefore V(x) = x(15 - 2x)(7 - 2x)$

$= 4x^3 - 44x^2 + 105x$

Differentiate: $V'(x) = 12x^2 - 88x + 105$

For stationary points, $V'(x) = 0$: $12x^2 - 88x + 105 = 0$

$(2x - 3)(6x - 35) = 0$

$\therefore x = 1\frac{1}{2}$ or $5\frac{5}{6}$

But x must be less than half the shortest side, i.e. $x < 3.5$, so we can disregard $x = 5\frac{5}{6}$. The only possible value for x is $x = 1.5$.

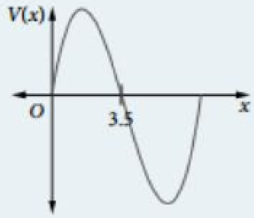
Use the first derivative test:

For $x < 1.5$, test $x = 1.4$: $V'(1.4) = 12 \times 1.4^2 - 88 \times 1.4 + 105 = 5.32 > 0$

For $x > 1.5$, test $x = 1.6$: $V'(1.6) = 12 \times 1.6^2 - 88 \times 1.6 + 105 = -5.08 < 0$

$V'(x)$ changes from positive to negative on passing through $x = 1.5$, so a maximum volume occurs when $x = 1.5$.

$V(1.5) = 1.5(15 - 3)(7 - 3) = 72\,\text{cm}^3$



A graph of $V(x)$ for $0 \le x \le 7$ helps to see what is happening.

Because $V(x) > 0$, the domain of the function is $0 < x < 3.5$.

The part of the graph below the x-axis is not relevant to the problem.

EXERCISE 14.5 PROBLEM SOLVING WITH DERIVATIVES

1 A rectangular block of land is enclosed by 160 m of fencing. If the breadth of the block is x m:

(a) express the length of the block in terms of x
(b) find the function $A(x)$ for the area of the block
(c) find the maximum area of the block that can be fenced using this fencing.

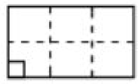
2 The sum of two numbers is 12. If one number is x, the value of x for which the product of the two numbers is a maximum is:

A 3 **B** 6 **C** 12 **D** 36

3 A rectangular block of land has one side along a river. The other three sides are to be fenced using 160 m of fencing. If the breadth of the block is x m:

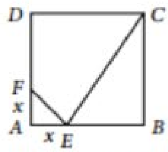
(a) express the length of the block in terms of x
(b) find the function $A(x)$ for the area of the block
(c) find the maximum area of the block that can be fenced using this fencing.

4 A rectangular paddock is to be fenced and also divided into six smaller rectangular paddocks, with one dividing fence parallel to the length and two dividing fences parallel to the breadth (as shown in the diagram by dashed lines). The total length of fencing to be used is 120 m. If the width of the paddock is x m and the breadth is y m, find:



(a) an expression giving y as a function of x
(b) the function $A(x)$ for the area of the original paddock
(c) the maximum possible area of the paddock.
(d) To allow access to the paddocks, six gates are to be added to the fences. Each gate is 3 m wide. What is the new maximum area of the large paddock?

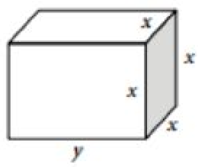
5 $ABCD$ is a square of unit length. Points E and F are on the sides AB and AD respectively so that $AE = AF = x$.



(a) Express the area of the quadrilateral $CDFE$ as a function of x.
(b) Find the greatest area that the quadrilateral can have.

6 A rectangular sheet of cardboard measures 16 cm by 6 cm. Equal squares are cut out of each corner and the sides are turned up to form an open rectangular box. What is the maximum volume of the box?

7 A block of wood in the shape of a cuboid has square ends of edge length x cm. The length of the block is y cm. The sum of length of the block and the perimeter of one end is 12 cm.



(a) Express y in terms of x.
(b) Find the volume V as a function of x.
(c) What is the largest possible volume of the block?

8 A box in the shape of a cuboid with a square base is to be made so that the sum of its dimensions $(l + b + h)$ is 20 cm. Find its maximum volume.

9 A rectangular box has a square base of edge length x cm. Its framework of 12 edges is constructed from wire of total length 36 cm. Find:

(a) the height of the box in terms of x **(b)** the volume of the box in terms of x

(c) the value of x for which the volume is a maximum.

10 A closed box in the shape of a cuboid has a total surface area of $216\,\text{cm}^2$ and a base length that is twice the width. If the width of the base is x cm, find:

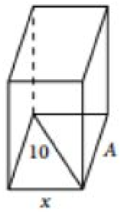
(a) the length of the base and the height in terms of x **(b)** the volume of the box in terms of x

(c) the maximum volume of the box.

11 A rectangular field is to be fenced along three sides using 300 m of fencing. The length of each equal end is x m and the length of the other side is y m. To find the dimensions of the field if its area is as large as possible, the following statements are made. Indicate whether each statement is correct or incorrect.

(a) $y = 300 - 2x$ **(b)** $A = 300x - 2x^2$ **(c)** $x = 75$ **(d)** Maximum area $= 11\,250\,\text{m}^2$

12 The diagonal of the base of a box in the shape of a cuboid has a length of 10 cm. One edge of the base has a length of x cm, as shown in the diagram.

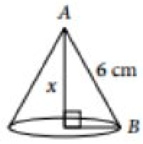


(a) Express, in terms of x, the length of the other edge of the base.

(b) The height of the box is equal to the length of this other edge. Find the volume of the box in terms of x.

(c) Calculate the maximum volume of the box.

13 The slant edge AB of a right circular cone is 6 cm. The vertical height of the cone is x cm, as shown in the diagram.



(a) Express the radius of the base in terms of x.

(b) Express the volume of the cone in terms of x.

(c) Find the vertical height of the cone when the volume is a maximum.

14 A piece of wire of length 30 cm is cut into two sections. Each section is then bent into the shape of a square. Find the smallest possible value of the sum of the areas of the two squares.

15 A block of metal is cast into the shape of a right cylinder with a total surface area of $20\pi\,\text{cm}^2$. The radius of the base is r cm and the height is h cm. The total surface area of a cylinder is given by $A = 2\pi r^2 + 2\pi rh$.

(a) Express h in terms of r. **(b)** Express the volume V in terms of r.

(c) Find the value of r for which the volume is greatest.

16 A piece of wire of length 50 cm is cut into two sections. One section is used to construct a rectangle whose dimensions are in the ratio 3 : 1; the other section is used to construct a square. Find the dimensions of the rectangle and the square so that the total enclosed area is a minimum.

17 Whale-watching boat trips go to sea with 20 or more passengers. For 20 passengers the charge is \$380 per person. For groups of more than 20, the price per person is reduced by \$12 for each additional person over 20 passengers.

(a) Show that the revenue function for the boat trip is given by $R = 620n - 12n^2$, where R is the revenue in dollars and n is the number of people in the group, $n \geq 20$.

(b) What number of passengers will produce the greatest revenue for the company?

18 Wobbly Skateboards looked at their sales in relation to their advertising budget. They found that the relationship between sales, $f(x)$, and thousands of dollars spent on advertising, x, was given by $f(x) = \frac{x^3}{3} - \frac{45x^2}{2} + 450x$, $10 \leq x \leq 40$.

(a) What number of advertising dollars can be expected to produce a maximum number of sales?

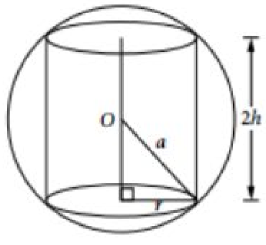
(b) What sales can be expected for that amount of advertising?

19 A company finds that the function $f(x) = x^3 - 96x^2 + 2880x$ provides a good approximation for their profit $f(x)$ in dollars, where x is the advertising expenditure in thousands of dollars.

(a) What expenditure on advertising would produce the maximum profit?

(b) What is this maximum profit?

20 A cylinder is inscribed in a sphere of radius a, centred at O. The height of the cylinder is $2h$ and the radius of the base is r, as shown in the diagram.



(a) Show that the volume V of the cylinder is given by $V = 2\pi r^2\sqrt{a^2 - r^2}$.

(b) Find the value of r for which the volume of the cylinder is a maximum. Explain why your value of r gives the maximum volume.

14.6 APPLICATIONS OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Example 15

For the function $f(t) = 2te^{-0.5t}$, find the value of t for which $f(t)$ has a maximum and hence calculate the maximum value. Sketch the graph of $f(t)$.

Solution

$f(t) = 2te^{-0.5t}$ Let $u = t$, $v = e^{-0.5t}$

$$f'(t) = 2\left(e^{-0.5t} + t \times \left(-\frac{1}{2}\right)e^{-0.5t}\right)$$

$$= e^{-0.5t}(2 - t)$$

For stationary points, $f'(t) = 0$: $e^{-0.5t}(2 - t) = 0$

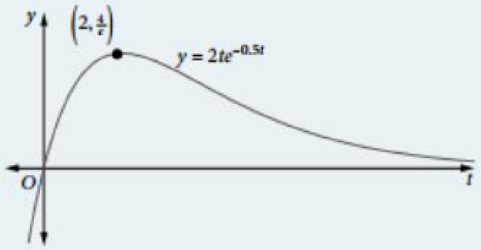
But $e^{-0.5t} > 0$ for all t, so $t = 2$ is the only solution and $f(2) = \frac{4}{e}$

For $t < 2$: $f'(t) > 0$

For $t > 2$: $f'(t) < 0$

Gradient changes from positive to negative as x increases, so $\left(2, \frac{4}{e}\right)$ is a maximum turning point.

The maximum value of the function is $\frac{4}{e} \approx 1.472$



$f(t) = 0$ at $t = 0$ as $e^{-0.5t} > 0$ for all t.

$t < 0$, $f(t) < 0$ $t < 2$, $f'(t) > 0$

$t > 0$, $f(t) > 0$ $t > 2$, $f'(t) < 0$

$t \to \infty$, $f(t) \to 0$ from above

$t = 0$ is a horizontal asymptote

Example 16

Find the coordinates of any maximum or minimum turning points of the curve $y = \dfrac{\ln x}{e^x}$ given that $x \ln x = 1$ when $x = 1.76$.

Solution

$$y = \frac{\ln x}{e^x} = \ln x \times e^{-x}$$

$$\frac{dy}{dx} = \ln x \times -e^{-x} + \frac{1}{x} \times e^{-x}$$

$$= \frac{1 - x\ln x}{xe^x}$$

For stationary points, $\frac{dy}{dx} = 0$:

$$\frac{1 - x\ln x}{xe^x} = 0$$

$$1 - x\ln x = 0$$

$$x\ln x = 1$$

$$\therefore x = 1.76$$

$x = 1.7$: $\quad \frac{dy}{dx} = \frac{1 - 1.7\ln 1.7}{1.7e^{1.7}} \approx 0.01 > 0$

$x = 1.8$: $\quad \frac{dy}{dx} = \frac{1 - 1.8\ln 1.8}{1.8e^{1.8}} = -0.005 < 0$

The gradient changes from positive to negative on passing through the stationary point so the function has a maximum value when $x = 1.76$.

$x = 1.76, y = 0.097$

The coordinates of the maximum turning point are (1.76, 0.097).

EXERCISE 14.6 APPLICATIONS OF THE EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1 Find the minimum value of $(x-2)e^x$.

2 Find the coordinates of the turning point of the curve $y = xe^{-0.5x}$ and state whether it is a maximum or minimum. Find the values of x for which:

(a) $y > 0$ **(b)** $\frac{dy}{dx} > 0$

3 Consider the function defined by the rule $f(x) = 3 - e^{-x}, x \geq 0$.

(a) Find the value of $f(0)$ and $f'(0)$. **(b)** Show that $f'(x) > 0$ for all values of x in the domain.

(c) What is the value of $\lim_{x\to\infty} f(x)$? **(d)** Sketch the graph of $f(x)$.

4 Consider the function defined by $f(x) = e^{-x^2}$ for all values of x.

(a) Find $f'(x)$.

(b) Find the values of x for which: **(i)** $f'(x) = 0$ **(ii)** $f'(x) > 0$ **(iii)** $f'(x) < 0$.

(c) Sketch the graph of the function.

5 The concentration of a certain drug in the blood at a time t hours after taking the dose is x units, where $x = 0.3te^{-1.1t}$.

(a) Determine the maximum concentration and the time at which this is reached.

(b) Plot the graph of $x = 0.3te^{-1.1t}$ for $t = 0, 0.1, 0.5, 1, 2, 3$ using graph paper or graphing software.

(c) This drug kills germs only while its concentration is at least 0.06 units. From the graph, find the length of time during which the drug will kill germs.

6 For $y = e^t + 4e^{-t}$, find the minimum value of y. Indicate whether each of the statements below is a correct or incorrect step in solving this problem.

(a) $y' = e^t - 4e^{-t}$ **(b)** Stationary point when $e^t = \pm 2$

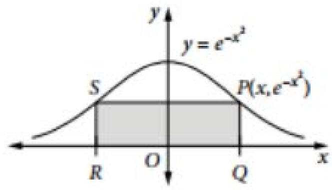
(c) $y'' = e^t + 4e^{-t}$ **(d)** Minimum value is 4

7 Sketch the graph of $f(t) = \dfrac{5}{2 + 3e^{-t}}$, $t \geq 0$.

(a) Show that $f'(t) > 0$ for all values of t in the domain. **(b)** Find $\lim\limits_{t \to \infty} f(t)$.

(c) State the range of the function.

8 The rectangle $PQRS$ has two vertices on the x-axis and two on the curve $y = e^{-x^2}$, as shown in the diagram. Find:



(a) the value of x for which the rectangle has a maximum area

(b) the maximum area of the rectangle.

9 Find the coordinates of any maximum or minimum turning points on the curve $y = \dfrac{\ln x}{x}$.

10 Show that $y = e^{2x} + 4e^{-2x}$ has a minimum value when $x = \dfrac{\ln 2}{2}$. What is the minimum value?

11 Find the minimum value of y if $y = x \ln x$ for $x > 0$.

12 **(a)** Find all values of x between 0 and 2π for which $\log_e (\sin x)$ is defined.

(b) Find the maximum value of $\log_e (\sin x)$ and when it occurs.

13 The sales revenue (in dollars) that a manufacturer receives for selling x units of a certain product can be approximated by the function $R(x) = 900 \log_e \left(1 + \dfrac{x}{300}\right)$.

Each unit costs the manufacturer \$1 to produce and the initial cost of adjusting the machinery for production is \$200, so that the total cost (in dollars) of the production of x units is $C(x) = 200 + x$.

(a) Find the profit, $P(x)$ dollars, obtained by the production and sale of x units and find the number of units which should be produced and sold for maximum profit. Calculate this maximum profit.

(b) Using technology, draw on the same diagram the graphs of $C(x)$ and $R(x)$ for $0 \leq x \leq 1500$. Show on the scale every 100 units on each axis. Use your graph to determine how many units must be produced to break even (i.e. where revenue first equals cost).

(c) If before being sold, the units are packaged in batches of ten, estimate from your graph minimum and maximum numbers of batches that can be produced and sold so that a profit is made.

(d) On the same diagram draw the graphs of $y = C(x)$, $y = R(x)$ and $y = P(x)$ and determine where $P(x) \geq 0$.

14 When a uniform chain is suspended at two fixed points, it hangs in a catenary whose equation is $y = \dfrac{1}{2a}\left(e^{ax} + e^{-ax}\right)$

(a) Sketch the curve when $a = 0.5$ and the fixed points are at the same horizontal level and 8 units apart.

(b) Find the sag at the centre.

(c) Find the angle of inclination of the chain at the supports.
(Using graphing software, set a slider for a and observe what happens as a changes.)

15 $f(x)$ is defined as $f(x) = e^{-x} \cos x$ in the domain $[0, \pi]$.

(a) Find $f(0)$, $f\left(\dfrac{\pi}{2}\right)$ and $f(\pi)$. **(b)** Find $f'(x)$.

(c) Evaluate $f'(0)$ and $f'\left(\dfrac{3\pi}{4}\right)$. **(d)** Sketch the graph of $y = f(x)$.

14.7 FURTHER APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

Example 17

Given the function $y = 3\sin t$, for $t > 0$:

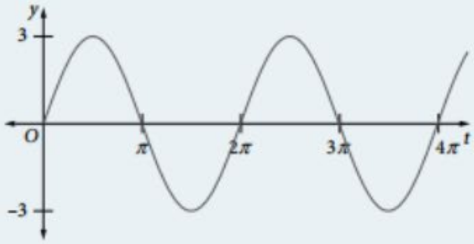
(a) Sketch the graph of this function for $0 \le t \le 4\pi$.

(b) Find the greatest and least values of the function and where they occur for $0 \le t \le 4\pi$.

(c) Describe the behaviour of the curve if y is the distance in metres to the right of a fixed point after a time t hours.

Solution

(a) $y = 3\sin t$, for $0 \le t \le 4\pi$:



(b) $\dfrac{dy}{dt} = 3\cos t$. For stationary points $\dfrac{dy}{dt} = 0$, so $\cos t = 0$.

For $0 \le t \le 4\pi$: $\quad t = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$

$\dfrac{d^2y}{dt^2} = -3\sin t$: $\quad$ at $t = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$ we have $\dfrac{d^2y}{dt^2} < 0$ $\quad \therefore$ maximum turning point

at $t = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$ we have $\dfrac{d^2y}{dt^2} > 0$ $\quad \therefore$ minimum turning point

At $t = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$, $y = 3$. The greatest value of the function is 3 and occurs at $t = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$.

At $t = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$, $y = -3$. The least value of the function is -3 and occurs at $t = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$.

(c) As t increases, y increases and then decreases in a repeating pattern. The function is periodic with a period of 2π hours (i.e. about 6 h 17 min). Its greatest distance from the fixed point is 3 metres in either direction. After moving 3 metres to the right, the object moves back through its starting point until it reaches a point 3 metres to the left; it then starts moving back to the right again until the pattern repeats.

The kind of wave-like repeating movement described in Example **17** is called **simple harmonic motion**. It is a type of motion that can be described using sine and cosine functions. It occurs naturally in the motion of pendulums and masses on springs, and it can also be used to model ocean tides and other wave-like movements.

EXERCISE 14.7 FURTHER APPLICATIONS OF TRIGONOMETRIC FUNCTIONS

1 Find the derivative of $\log_e(\cos x)$.

2 Find the equation of the tangent to the curve $y = \tan x$ at $x = \dfrac{\pi}{4}$.

3 For $f(x) = \sin x + \cos x$ over the domain $0 \le x \le 2\pi$, find:

(a) $f'(x)$ **(b)** $f''(x)$ **(c)** the coordinates of any turning points
(d) the coordinates of any points of inflection **(e)** the maximum value of $f(x)$.

4 For $y = e^{\sin x}$, find the equation of the normal to the curve at the point where $x = 0$.

5 Find all the points on the graph of $y = 2\sin x + \sin^2 x$, $0 \le x \le 4\pi$, at which the tangent is horizontal.

6 The tide at a point on the WA coast can be modelled using the equation $y = a\cos nt$. At Cable Beach in WA, over two consecutive days, the average difference between high and low tides is 9.0 metres and the average time between high tide and low tide is 6.1 hours.

(a) What is the amplitude of the tide function at Cable Beach?
(b) How much time passes between successive high tides (i.e. the period) and what is the value of n?
(c) Use this information to obtain the tide function and draw its graph.
(d) If the depth of water at low tide is 0.5 metres, what is the depth of the water 1 hour after low tide?

7 Consider the function $y = \sin x + \cos x$ for $0 < x < 2\pi$.

(a) For what values of x is $\frac{dy}{dx} = 0$?
(b) Find the greatest and least values of y and when they occur.
(c) For what values of x is $\frac{dy}{dx} > 0$?
(d) Find the coordinates of any points of inflection.

8 Find the equation of the normal to the curve $y = \cot x$ at the point $P\left(\frac{\pi}{4}, 1\right)$.

9 If $y = e^{\operatorname{cosec} x}$, find the equation of the tangent to the curve at $x = \frac{\pi}{2}$.

10 If $y = 3\cos 4x$, prove that $\frac{d^2y}{dx^2} + 16y = 0$.

11 The population of rock wallabies on an island is given by $P(t) = 200 + 40\cos\left(\frac{\pi}{6}t\right)$, where t is the time in months after the population was first measured.

(a) Find all times during the first 12 months when the population is 180 rock wallabies.
(b) Sketch the graph of $P(t)$ for $0 \le t \le 12$.
(c) When is $\frac{dP(t)}{dt} > 0$?
(d) What is the greatest population of rock wallabies over this twelve-month period?
(e) Between what values does the population of rock wallabies range over this twelve-month period?

14.8 USING DERIVATIVES IN MOTION IN A STRAIGHT LINE

In Chapter 7 you met the terms displacement, velocity and acceleration. They were linked to the derivative of a function.

Displacement

Displacement is defined as the position relative to a starting point. It can be positive or negative. Displacement does not necessarily represent the total distance travelled.

Unlike displacement, distance is always a positive quantity.

Velocity

Velocity is defined as the rate of change of position (i.e. of displacement) with respect to time, or as the time rate of change of position in a given direction.

$$v(t) = f'(t) = \frac{dx}{dy} = \dot{x} = \lim_{h \to 0} \frac{f(t+h) - f(t)}{h}$$

Velocity can be positive or negative, depending on the direction of travel.

Speed is the magnitude of the velocity and is always positive.

Acceleration

Acceleration is defined as the rate of change of velocity with respect to time. Acceleration, like velocity, can be positive or negative. Positive acceleration indicates that the velocity is increasing, while negative acceleration indicates that the velocity is decreasing, which is often called deceleration or retardation.

(Note that 'increasing velocity' is not necessarily 'faster speed'; it only means acceleration in the direction of positive displacement.)

If you denote the velocity by $v(t)$, then the average acceleration over the interval from t to $(t + h)$ is $\frac{v(t+h)-v(t)}{h}$.

The instantaneous acceleration at time t is defined by $\lim_{h\to 0}\frac{v(t+h)-v(t)}{h}$. It may be denoted by

$v'(t)$, $a(t)$, $f''(t)$, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$, or $\ddot{x}$: $a(t) = v'(t) = \frac{d^2x}{dt^2} = \ddot{x} = \lim_{h\to 0}\frac{v(t+h)-v(t)}{h}$

Summary of important terms

'initially': $t = 0$ 'at the origin': $x = 0$

'at rest': $v = 0$ 'velocity is constant': $a = 0$

Units and symbols

Physical quantity	Unit	Symbol
Time	s	t
Displacement	cm, m	x (or s in physics)
Velocity	cm s^{-1}, m s^{-1}	v, $\frac{dx}{dt}$, $\dot{x}$
Acceleration	cm s^{-2}, m s^{-2}	a, $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$ $\ddot{x}$

Note that 's' is the abbreviation for second, 'cm' for centimetre and 'm' for metre.
Constant acceleration due to gravity = $9.8\ \text{m s}^{-2}$.

Example 18

A ball is projected vertically upwards from the top of a building 30 metres high. The equation for its motion is given by $x = 30 + 25t - 5t^2$, where x is the displacement in metres above the top of the building and t is in seconds.

(a) Graph the displacement function.

(b) Find the velocity as a function of time.

(c) What is the initial velocity of the ball?

(d) The ball reaches its greatest height when $\frac{dx}{dt} = 0$. When does it reach its greatest height and how high above the ground is it then?

(e) How long will it take for the ball to hit the ground?

(f) What is the ball's speed when it hits the ground?

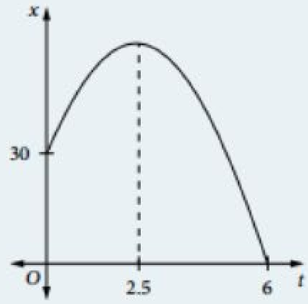
(g) Find the expression for the acceleration of the ball.

Solution

(a) $x = 30 + 25t - 5t^2$

The graph will be a parabola so the axis of symmetry is given by $t = \frac{-25}{2\times(-5)} = 2.5$

t	0	1	2	2.5	3	4	5	6
x	30	50	60	61.25	60	50	30	0



(b) $x = 30 + 25t - 5t^2$: $v = \frac{dx}{dt} = 25 - 10t$

(c) Initial velocity at $t = 0$: $v = 25\,\text{m}\,\text{s}^{-1}$

(d) $\frac{dx}{dt} = 0$: $25 - 10t = 0$

$t = 2.5$ seconds (This could have been obtained from the graph).

$t = 2.5$: $x = 61.25$ m

The highest point is 61.25 metres above the ground, which the ball reaches after 2.5 seconds.

(e) The ball reaches the ground when $x = 0$: using the graph the answer is 6 seconds.

Using the displacement function: $30 + 25t - 5t^2 = 0$

$$5(6 + 5t - t^2) = 0$$
$$(6 - t)(1 + t) = 0$$

As $t \geq 0$, $t = 6$.

The ball hits the ground after 6 seconds.

(f) $t = 6$: $v = 25 - 60 = -35$

The ball hits the ground with a speed of $35\,\text{m}\,\text{s}^{-1}$.
As the initial upwards velocity is positive, the velocity when it hits the ground is negative as the ball is moving downwards.

(g) $a = \frac{dv}{dt} = -10\,\text{m}\,\text{s}^{-2}$

This means that the acceleration is acting in the opposite direction to the initial upwards velocity.

EXERCISE 14.8 USING DERIVATIVES IN MOTION IN A STRAIGHT LINE

1 A particle is moving in a straight line so that its displacement x metres is given by $x = \frac{t^3}{2} - 3t^2 + 5$.

(a) Find an expression for its velocity. **(b)** Find an expression for its acceleration.

(c) When is the velocity zero? **(d)** Find the displacement, velocity and acceleration after 4 seconds.

2 The displacement x metres at time t seconds, $t \geq 0$, of a particle moving in a straight line is given by $x = 4t^3 - 3t^2 + 5t - 1$. Its acceleration is given by:

A $a = 4t^3 - 3t^2 + 5t - 1$ **B** $a = 12t^2 - 6t + 5$ **C** $a = 24t - 6$ **D** $a = 24$

3 The displacement x metres at time t seconds, $t \geq 0$, of a particle moving in a straight line is given by $x = 2t^3 - 6t^2 - 30t$.

(a) Find the velocity and acceleration at any time t.

(b) Find the initial velocity and acceleration.

(c) At what time is the velocity zero? What is the acceleration at this time?

(d) During what time interval is the velocity negative?

4 A particle is projected vertically upwards from the ground. The equation for its motion is given by $x = 30t - 5t^2$, where x is the displacement in metres above the ground and t is in seconds.

(a) Graph the displacement function.
(b) Find the velocity as a function of time.
(c) What is the initial velocity of the particle?
(d) When does the particle reach its greatest height and how high above the ground is it then?
(e) How long will it take before the particle returns to the ground?
(f) What is the particle's speed when it hits the ground?
(g) Find the expression for the acceleration of the particle.

5 The velocity of a function is given by $v = 5e^{-t}$.

(a) Find the expression for a, the acceleration.
(b) Explain why v is a decreasing function.
(c) The displacement function is not given. Without finding the displacement function, determine whether it has any stationary points. Justify your answer.

6 An object moves with a velocity v given by $v = 20 + (2t - 1)e^{-0.5t}$, where t is in hours and v is in km h^{-1}. Calculate:

(a) the velocity after 1 hour
(b) the time taken to reach its maximum velocity.

CHAPTER REVIEW 14

1 For the graph of $y = 15x + 12x^2 - 4x^3$ for $-1 \le x \le 3$, find the values of x for which:

(a) y increases as x increases
(b) y decreases as x increases
(c) y is a maximum
(d) y is a minimum.

2 If $f(x) = 2x^3 + 3x^2 - 12x$, find the values of x for which: (a) $f'(x) = 0$ (b) $f'(x) > 0$ (c) $f'(x) < 0$

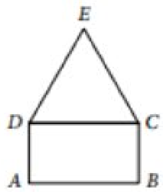
3 Sketch the graph of $y = f(x)$, given that:

(a) $f(3) = 5$, $f'(3) = 0$, $f'(x) > 0$ for $x < 3$ and $f'(x) < 0$ for $x > 3$
(b) $f(-1) = 8$, $f'(-1) = 0$, $f(2) = 3$, $f'(2) = 0$, $f'(x) < 0$ for $-1 < x < 2$, and $f'(x) > 0$ for $x < -1$ and for $x > 2$.

4 Find the coordinates of the points on the following curves where the gradient is zero. Determine whether these points are local maximum or minimum points.

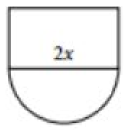
(a) $y = 3x^3 - 2x^2$
(b) $y = x^3 - 3x^2 - 9x$

5 A figure $ABCED$ consists of a rectangle $ABCD$ topped by an equilateral triangle CED as shown in the diagram. If the perimeter of the figure is 45 cm, find the dimensions of the rectangle when the total area is a maximum.



6 A piece of wire 8 m long is cut into two parts. One part is bent into the shape of a square and the other part is bent into a rectangle whose length is twice its breadth. Calculate the length of each part if the sum of the areas of the square and the rectangle is to be a minimum.

7 A figure consists of a semicircle with a rectangle constructed on its diameter, as shown in the diagram. If the perimeter of the figure is 50 cm, find the dimensions of the rectangle such that the area of the figure is as large as possible. What is this largest area?



8 For the graph of $y = f(x)$ where $f(x) = \dfrac{x^3}{3} - 4x + 3$, find the following:

(a) the values of x for which $f'(x) = 0$
(b) the values of x for which $f'(x) < 0$
(c) any local maximum and minimum values of $f(x)$.
(d) Sketch the curve.

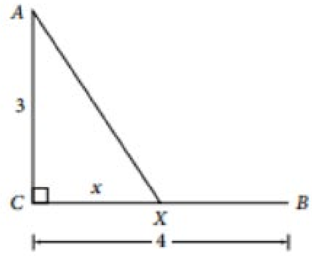
9 Sketch the graph of $y = 3x^3 - 5x^2$ for values of x in the domain $-0.5 \le x \le 1.5$, locating the turning points.

10 A rectangular sheet of metal measures 6 cm by 4 cm. Four equal squares are cut out of the corners and the sides are turned up to form an open rectangular box. Find the edge length of the squares cut so that the box has a maximum volume.

11 Prove that the curve $y = x^2(3 - x)$ has a horizontal tangent where $x = 2$ and crosses the y-axis at right angles at the origin.

12 Sketch the graph of $y = x^3(3 - x)$ in the domain $-1 \le x \le 3$, giving the coordinates of the turning points and points of inflection.

13 Jack is in the bush at point A, 3 km from the nearest point C, which is at one end of a straight 4 km path CB, as shown in the diagram. Jack wants to get to point B, the other end of the path, as quickly as possible. He can run at a speed of $20\,\text{km}\,\text{h}^{-1}$ along the path CB but only at $10\sqrt{2}\,\text{km}\,\text{h}^{-1}$ in the bush off the path. He runs in a straight line through the bush from A to a point X on the path CB, then along the path from X to B.



(a) Find, in terms of x, the time taken for Jack to go from:
  (i) A to X **(ii)** X to B.

(b) Find, in terms of x, the total time t hours to get from A to B.

(c) Find the position of the point X for which t is a minimum. Find this minimum time.

14 A haulage company makes frequent deliveries from Sydney to Cairns and calculates that the overhead cost \$$C$ depends on the average delivery speed $v\,\text{km}\,\text{h}^{-1}$ according to the rule $C = v + \frac{3600}{v}$. Find the average delivery speed to minimise the overhead cost.

15 **(a)** Find the maximum value of $2xe^{-1.5x}$ and the value for which this function has a maximum value.
(b) If $f(x) = 2xe^{-1.5x}$, find $f(0)$, $f(0.5)$, $f(1)$ and hence graph the function in the domain $0 \le x \le 1$.

16 If $\theta = \theta_0 e^{-kt}$, show that $\frac{d\theta}{dt} = -k\theta$.

17 A car is worth \$10 000 when new. After t years, the value of the car (in dollars) is given by the formula $V = Ae^{-0.2t}$.

(a) Find the value of A. **(b)** Find the value of the car after 6 years.

(c) Find the rate in dollars per year at which the car's value is depreciating, when:
  (i) $t = 6$ **(ii)** $V = 5000$

(d) How much time will it take until the car is worth only \$1000?

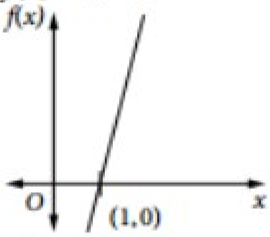
18 Find the turning points of $y = 3\sec 2x$ for $-\frac{\pi}{4} < x < \frac{3\pi}{4}$ and determine their nature.

CHAPTER 14

EXERCISE 14.1

1 A, C

2 $f(x) = 2x - 2$



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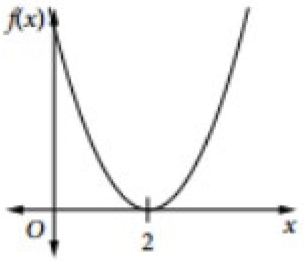

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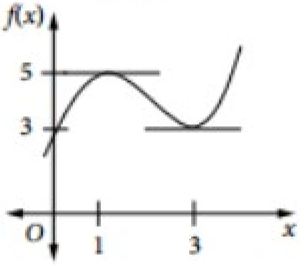

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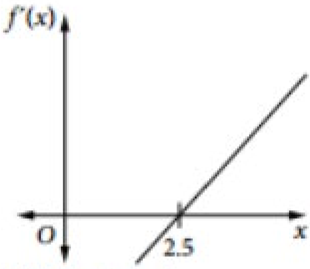
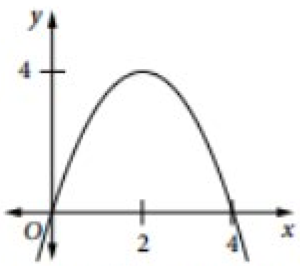


$f'(x) = 2x - 5$

(a) $x < 2.5$ (b) $x = 2.5$

(c) $x > 2.5$

6



$x = 2$; positive, negative

7 (a) $x < -1.5$ (b) $x > -1.5$ (c) $x = -1.5$

8 (a) $f'(x) = 3x^2 - 12x + 9$ (b) $x < 1, x > 3$

(c) $1 < x < 3$ (d) $x = 1$

9 (a) real x (b) none (c) never

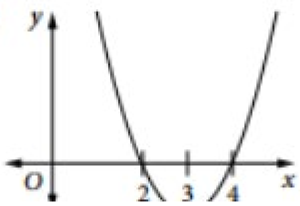
10 (a) $x = -\frac{1}{3}, 1$ (b) $x < -\frac{1}{3}, x > 1$ (c) $-\frac{1}{3} < x < 1$

EXERCISE 14.2

1 (a) $f'(x) = 2x - 6$

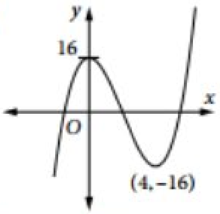
(b) $(3, -1)$ minimum turning point

(c)



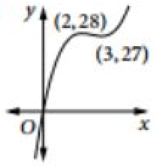
2 C

3 (a) $f'(x) = 3x^2 - 12x$
(b) $(0, 16)$ local maximum; $(4, -16)$ local minimum
(c)

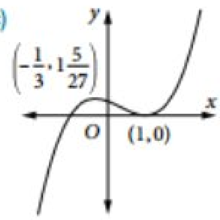


4 (a) correct (b) incorrect (c) correct (d) correct

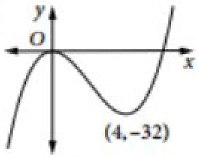
5 (a) $f'(x) = 6x^2 - 30x + 36$
(b) $(2, 28)$ local maximum; $(3, 27)$ local minimum
(c)



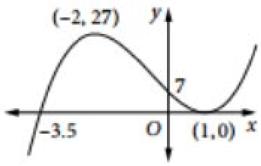
6 (a) $f'(x) = 3x^2 - 2x - 1$
(b) $-\frac{1}{3}, 1\frac{5}{27}$ local maximum; $(1, 0)$ local minimum
(c)



7 $3\frac{1}{8}$ 8 2 9



10 $(-2, 27)$ maximum; $(1, 0)$ minimum



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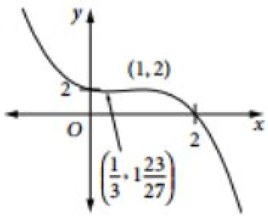
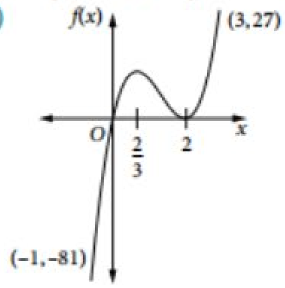


$y' = -(3x^2 - 4x + 1)$:
$\frac{1}{3}, 1\frac{23}{27}$ minimum,
$(1, 2)$ maximum

12 (a) $x = \frac{2}{3}, 2$ (b) $x < \frac{2}{3}, x > 2$ (c) $\frac{2}{3} < x < 2$
(d)



range = 108
greatest value = 27
least value = −81

13 $y' = 2ax + b$: $y' = 0$ where $2ax + b = 0$, $x = \frac{-b}{2a}$;
$y'' = 2a$ so the turning point is minimum if $a > 0$, maximum if $a < 0$

14 $\frac{dy}{dx} = -\frac{1}{x^2}$: $\frac{dy}{dx}$ is never zero, hence no stationary points, hence no turning points; $\frac{dy}{dx} < 0$ for all x in the domain

EXERCISE 14.3

1 (a) 6 (b) $6x + 4$ (c) -2 (d) $20x^3 + 12x$ (e) $-12x^2 + 4$ (f) 0

2 D

3 (a) correct (b) incorrect (c) correct (d) incorrect

4 (a) $\frac{-1}{4x\sqrt{x}}$ (b) $\frac{-1}{4(x-2)\sqrt{x-2}}$ (c) $\frac{x(2x^2+3)}{(x^2+1)^{\frac{3}{2}}}$
(d) $\frac{2}{x^3}$ (e) $\frac{2}{(x+1)^3}$ (f) $\frac{-6}{(x+3)^3}$
(g) $\frac{3(x^2+1)}{4x^2\sqrt{x}}$ (h) $\frac{15x^2+1}{4x\sqrt{x}}$ (i) $\frac{3x^2-18x+11}{4(x-1)^{\frac{3}{2}}(x+1)^3}$

5 real x 6 real x 7 (a) $x > -2$ (b) $x < -2$ (c) $(-2, 22)$

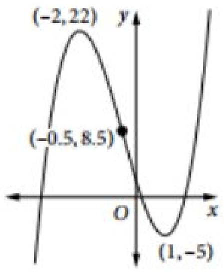
8 $x > -1$
$\frac{d^2y}{dx^2} \neq 0$ for x in the domain $\therefore$ no point of inflection

9 (a) $x > 0$ (b) $x < 0$
(c) $\frac{d^2y}{dx^2} \neq 0$ for x in the domain $\therefore$ no point of inflection

10 $\frac{d^2y}{dx^2} = \frac{6}{x^4}$; $\frac{d^2y}{dx^2} > 0$ for x in the domain $\therefore$ concave up

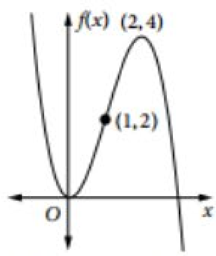
EXERCISE 14.4

1 $(-2, 22)$ maximum; $(1, -5)$ minimum;

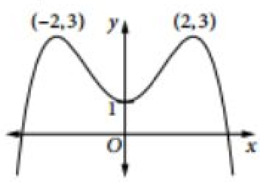


2 A

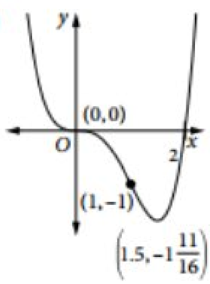
3 $(0, 0)$ minimum; $(2, 4)$ maximum; $(1, 2)$ inflection



4 $(0, 1)$ min; $(2, 3)$ max; $(-2, 3)$ max; greatest value of function is 3



5 (a) $\left(1\frac{1}{2}, -1\frac{11}{16}\right)$ minimum
(b) $(0, 0)$ horizontal inflection, $(1, -1)$ inflection
(c)



(d) $-1\frac{11}{16}$

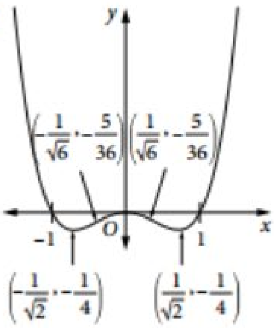
6 (a) $x=2$ (b) $x>2$ (c) $x<2$

7 $(-1,-9),(2,-48)$; $x<-1, x>2$

8 (a) $(-1,0),(0,0),(1,0)$

(b) $(0,0)$ max; $\left(\pm\frac{1}{\sqrt{2}},-\frac{1}{4}\right)$ min (c) $\left(\pm\frac{1}{\sqrt{6}},-\frac{5}{36}\right)$

(d)



(e) $-\frac{\sqrt{6}}{6}<x<\frac{\sqrt{6}}{6}$

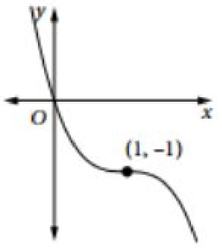
9 (a) correct (b) correct (c) incorrect (d) correct

10 $\frac{dy}{dx}=-3(x-1)^2<0$ for all $x\neq 1$

$\frac{d^2y}{dx^2}=-6(x-1)=0$ where $x=1$;

concavity changes, so $(1,-1)$ is a point of inflection.

$y=-x(x^2-3x+3)$, $\Delta=9-12<0$, $x^2-3x+3=0$ has no real roots; the curve only cuts the x-axis at $(1,-1)$



11 (a) $(1,0)$ minimum

(b) $y''=12x^2$: $y''=0$ at $x=0$ but concavity does not change

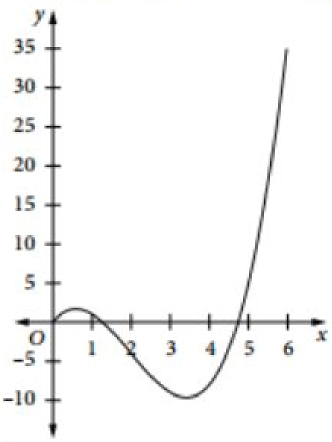
$\therefore$ no point of inflection, curve is always concave up

(c) Global minimum is 0

12 (a) Greatest value of the function is 36 when $x=6$.

Least value of the function is $-4\left(1+\sqrt{2}\right)$ when $x=2+\sqrt{2}$.

(b)



13 \$4.50

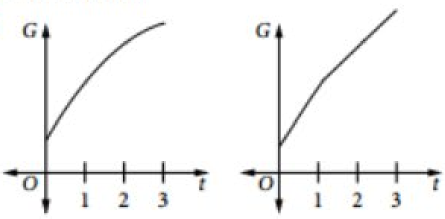
14 40 units

15 (a) 90 (b) \$17420 (c) $P=374x-2.2x^2-400$ (d) 85, \$15495

16 (a) decreasing (b) it is increasing at a decreasing rate

(c) concave down

(d)



(e) $f'(t)=3.2$, point of inflection

(f) it is increasing at a decreasing rate

EXERCISE 14.5

1 (a) $l=80-x$ (b) $A(x)=x(80-x)=80x-x^2$

(c) $1600\,\text{m}^2$

2 B

3 (a) $l=160-2x$ (b) $A(x)=x(160-2x)$ (c) $3200\,\text{m}^2$

4 (a) $y=\frac{120-4x}{3}$ (b) $A(x)=\frac{x(120-4x)}{3}$

(c) $x=15, y=20, A=300\,\text{m}^2$ (d) $396.75\,\text{m}^2$

5 (a) $A=\frac{1+x-x^2}{2}$ (b) $\frac{5}{8}$ units squared

6 $\frac{1600}{27}\,\text{cm}^3$

7 (a) $y=12-4x, 0<x<3$ (b) $V=12x^2-4x^3$ (c) $16\,\text{cm}^2$

8 $\frac{8000}{27}\,\text{cm}^3$

9 (a) $h=9-2x, 0<x<4.5$ (b) $V=x^2(9-2x)$ (c) $x=3$

10 (a) length $=2x$, height $=\frac{108-2x^2}{3x}$

(b) $V=\frac{2x\left(108-2x^2\right)}{3}$ (c) $144\sqrt{2}\,\text{cm}^3$

11 (a) correct (b) correct (c) correct (d) correct

12 (a) $y=\sqrt{100-x^2}$ (b) $V=100x-x^3$ (c) $\frac{2000\sqrt{3}}{9}\,\text{cm}^3$

13 (a) $r=\sqrt{36-x^2}$ (b) $V=\frac{\pi x\left(36-x^2\right)}{3}$ (c) $2\sqrt{3}$ cm

14 $28\frac{1}{8}\,\text{cm}^2$

15 (a) $h=\frac{10-r^2}{r}$ (b) $V=\pi(10r-r^3)$ (c) $r=\frac{\sqrt{30}}{3}$

16 rectangle $3\frac{4}{7}$ cm $\times$ $10\frac{5}{7}$ cm; square sides $5\frac{5}{14}$ cm

17 (a) $R=(380-12(n-20))\times n=620n-12n^2$

(b) 26 passengers

18 (a) \$15000 (b) 2812 or 2813

19 (a) \$24000 (b) \$27648

20 (a) $h^2+r^2=a^2$, $h=\sqrt{a^2-r^2}$, $V=\pi r^2\times 2h=2\pi r^2\sqrt{a^2-r^2}$

(b) $V'(r)=2\pi r\left(2a^2-3r^2\right)$;

$3r^2=2a^2$, $r=\frac{a\sqrt{6}}{3}$

$V''(r)=2\pi(2a^2-9r^2)$

$V''\left(\frac{a\sqrt{6}}{3}\right)=2\pi\left(2a^2-\frac{\cancel{9}\times 6a^2}{\cancel{9}}\right)<0$;

$r=\frac{a\sqrt{6}}{3}$ gives max volume

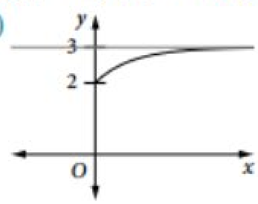
EXERCISE 14.6

1 $-e$

2 $\frac{dy}{dx}=\frac{(2-x)e^{-0.5x}}{2}$; $\left(2,\frac{2}{e}\right)$, maximum (a) $x>0$ (b) $x<2$

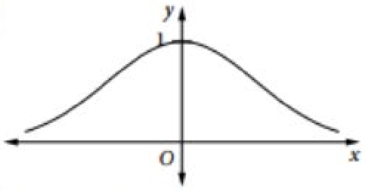
3 (a) $f(0)=2$, $f'(0)=1$ (b) $f'(x)=e^{-x}>0$ for all x (c) 3

(d)



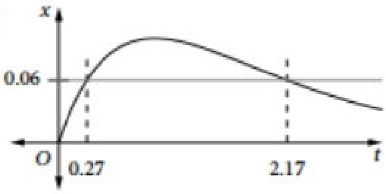
4 (a) $f'(x) = -2xe^{-x^2}$ (b) (i) $x = 0$ (ii) $x < 0$ (iii) $x > 0$

(c)



5 (a) 0.1 units after 0.91 hours

(b)



(c) 2.00 hours

6 (a) correct (b) incorrect (c) correct (d) correct

7

(a) $f'(t) = \dfrac{-5\times\left(-3e^{-t}\right)}{\left(2+3e^{-t}\right)^2} = \dfrac{15e^{-t}}{\left(2+3e^{-t}\right)^2} > 0$ because $e^{-t} > 0$ and $\left(2+3e^{-t}\right)^2 > 0$ for all t

(b) 2.5 (c) $1 \le f(t) < 2.5$

8 (a) $x = \dfrac{1}{\sqrt{2}}$ (b) $\sqrt{2}e^{-0.5} \approx 0.86$ units^2

9 $\dfrac{dy}{dx} = \dfrac{1-\ln x}{x^2}$. Stationary points: $x = e$, $y = \dfrac{1}{e}$.

$\dfrac{d^2y}{dx^2} = \dfrac{2\ln x - 3}{x^3} < 0$ when $x = e$. Maximum at $\left(e, \dfrac{1}{e}\right)$.

10 $\dfrac{dy}{dx} = 2e^{2x} - 8e^{-2x}$. Stationary points: $x = \dfrac{\ln 2}{2}$, $y = 4$.

$\dfrac{d^2y}{dx^2} = 16 > 0$. Minimum value is 4 when $x = \dfrac{\ln 2}{2}$.

11 $\dfrac{dy}{dx} = 1 + \ln x$. Stationary points: $x = \dfrac{1}{e}$, $y = -\dfrac{1}{e}$. $\dfrac{d^2y}{dx^2} = e > 0$.

Minimum when $x = \dfrac{1}{e}$ is $-\dfrac{1}{e}$.

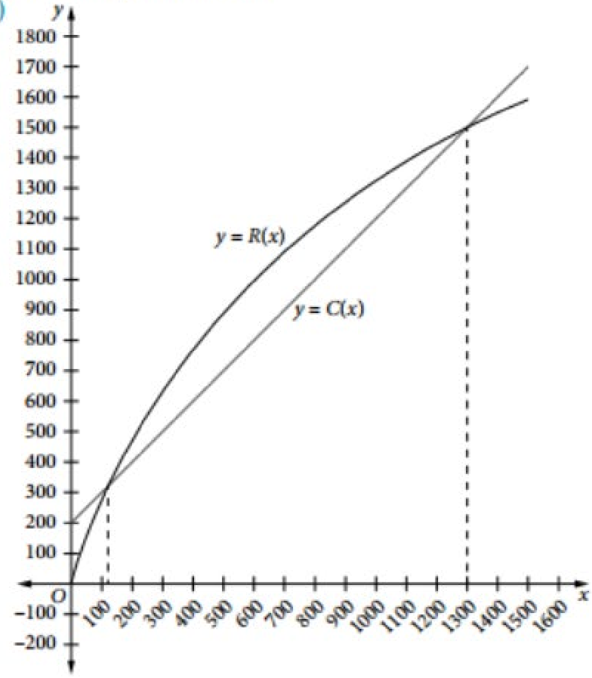
12 (a) Require $\sin x > 0$: $0 < x < \pi$.

(b) $\dfrac{dy}{dx} = \cot x$. Stationary points: $x = \dfrac{\pi}{2}$, $y = 0$.

$\dfrac{d^2y}{dx^2} = -\operatorname{cosec}^2 x = -1 < 0$. Maximum when $x = \dfrac{\pi}{2}$ is 0.

13 (a) Maximum profit is \$188.75

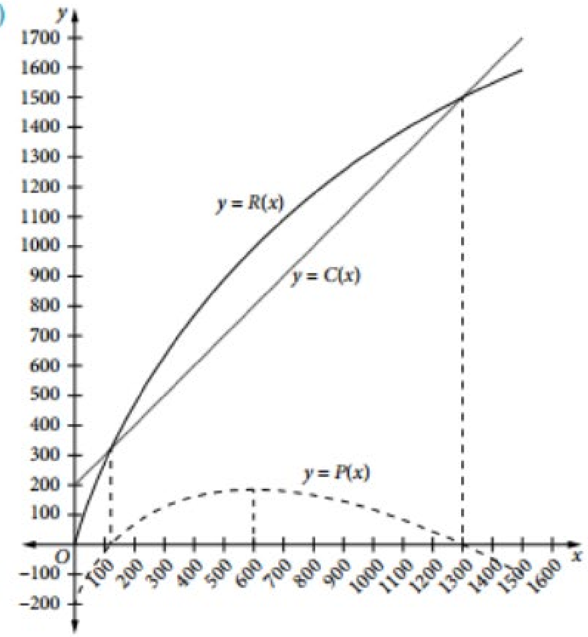
(b)



First breaks even when $x \approx 130$

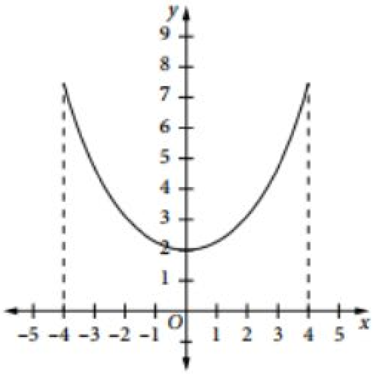
(c) A profit is made when the revenue > cost. This is between about 130 and 1300, or 13 batches of ten and 130 batches of ten.

(d)



$P(x) \ge 0$ for $130 \le x \le 1300$

14 (a) $a = 0.5,\ y = e^{\frac{x}{2}} + e^{-\frac{x}{2}}$



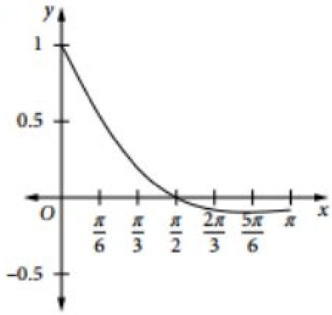
(b) Least height of function is 2 units so sag = 5.52 units.
(c) Angle of inclination at the ends is 74° 35′.

15 (a) $f(0) = 1,\ f\left(\frac{\pi}{2}\right) = 0,\ f(\pi) = -e^{-\pi}$

(b) $f'(x) = -e^{-x}(\sin x + \cos x)$

(c) $f'(0) = -1,\ f'\left(\frac{3\pi}{4}\right) = -e^{-\frac{3\pi}{4}}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = 0$

(d)



EXERCISE 14.7

1 $-\tan x$

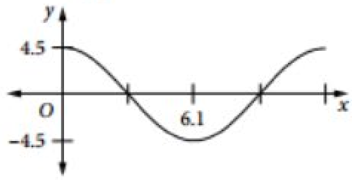
2 $4x - 2y + 2 - \pi = 0$

3 (a) $\cos x - \sin x$ (b) $-\sin x - \cos x$ (c) $\left(\frac{\pi}{4}, \sqrt{2}\right), \left(\frac{5\pi}{4}, -\sqrt{2}\right)$
(d) $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$ (e) $\sqrt{2}$

4 $y = 1 - x$

5 $\frac{dy}{dx} = 2\cos x + 2\sin x\cos x = 2\cos x(1 + \sin x)$; $\left(\frac{\pi}{2}, 3\right), \left(\frac{3\pi}{2}, -1\right)$,
$\left(\frac{5\pi}{2}, 3\right), \left(\frac{7\pi}{2}, -1\right)$

6 (a) 4.5 m (b) 12.2 hours; $n = \frac{10\pi}{61}$
(c) $y = 4.5\cos\frac{10\pi t}{61}$



(d) 1.08 m

7 (a) $\frac{dy}{dx} = \cos x - \sin x$. $\frac{dy}{dx} = 0$ when $\tan x = 1$. $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

(b) $x = \frac{\pi}{4}, y = \sqrt{2}$. $x = \frac{5\pi}{4}, y = -\sqrt{2}$. $\frac{d^2y}{dx^2} = -y$.

$x = \frac{\pi}{4}$: $\frac{d^2y}{dx^2} < 0$. Maximum at $\left(\frac{\pi}{4}, \sqrt{2}\right)$.

$x = \frac{5\pi}{4}$: $\frac{d^2y}{dx^2} > 0$. Minimum at $\left(\frac{5\pi}{4}, -\sqrt{2}\right)$.

Greatest value of y is $\sqrt{2}$ at $x = \frac{\pi}{4}$ and the least value is $-\sqrt{2}$ at $x = \frac{5\pi}{4}$.

(c) $\frac{dy}{dx} > 0$ for $0 < x < \frac{\pi}{4}$ and $\frac{5\pi}{4} < x < 2\pi$.

(d) $\frac{d^2y}{dx^2} = 0$: $\tan x = -1$, $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. Points of inflection at $\left(\frac{3\pi}{4}, 0\right), \left(\frac{7\pi}{4}, 0\right)$.

8 $\frac{dy}{dx} = -\operatorname{cosec}^2 x$. $P\left(\frac{\pi}{4}, 1\right)$: $\frac{dy}{dx} = -2$. Gradient of normal $= \frac{1}{2}$.

Equation of normal: $y - 1 = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$ or $4x - 8y + 8 - \pi = 0$

9 $\frac{dy}{dx} = e^{\operatorname{cosec} x}(-\operatorname{cosec} x \cot x)$. $x = \frac{\pi}{2}$: $\frac{dy}{dx} = 0, y = e$.

The equation of the tangent is $y = e$.

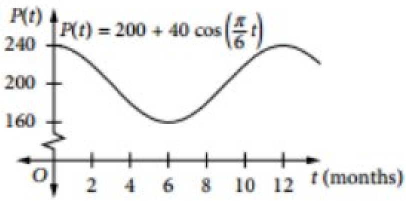
10 $y = 3\cos 4x$. $\frac{dy}{dx} = -12\sin 4x$. $\frac{d^2y}{dx^2} = -48\cos 4x$.

$\text{LHS} = \frac{d^2y}{dx^2} + 16y = -48\cos 4x + 16 \times 3\cos 4x$
$= -48\cos 4x + 48\cos 4x = 0 = \text{RHS}$

11 (a) $180 = 200 + 40\cos\left(\frac{\pi}{6}t\right)$. $\cos\left(\frac{\pi}{6}t\right) = -\frac{1}{2}$. $t = 4, 8$.

After 4 months and 8 months.

(b)



(c) $\frac{dP(t)}{dt} = -\frac{20\pi}{3}\sin\left(\frac{\pi}{6}t\right)$.

$\frac{dP(t)}{dt} > 0$: $\sin\left(\frac{\pi}{6}t\right) < 0$, $6 < t < 12$.

From the graph: between 6 and 12 months.

(d) The greatest population is 240 wallabies.

(e) From 160 to 240 wallabies.

EXERCISE 14.8

1 $x = \frac{t^3}{2} - 3t^2 + 5$

(a) $v = \frac{3t^2}{2} - 6t$ (b) $a = 3t - 6$

(c) $\frac{3t^2}{2} - 3t = 0$, $\frac{3t}{2}(t - 4) = 0$, $t = 0, 4$

(d) $t = 4$: $x = -11, v = 0, a = 6$

2 C

3 $x = 2t^3 - 6t^2 - 30t$.

(a) $v = 6t^2 - 12t - 30$, $a = 12t - 12$

(b) $t = 0$: $v = -30$, $a = -12$

(c) $v = 0$: $6t^2 - 12t - 30 = 0$, $6(t^2 - 2t - 5) = 0$,

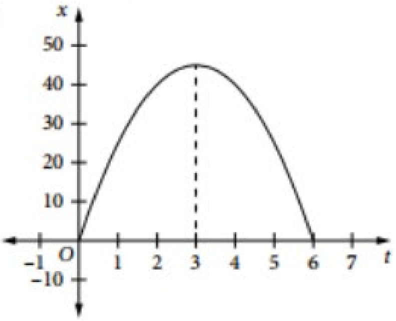
$t = \dfrac{2 \pm \sqrt{4 + 20}}{2} = 1 \pm \sqrt{6}$,

$t = 1 + \sqrt{6}$

$a = 12(1 + \sqrt{6}) - 12 = 12\sqrt{6}$

(d) $t^2 - 2t - 5 < 0$, $0 \le t < 1 + \sqrt{6}$

4 (a)



(b) $v = 30 - 10t$. (c) $v = 30\ \text{m s}^{-1}$

(d) $v = 0$: $t = 3$ s, $x = 90 - 45 = 45$ m (e) 6 seconds

(f) $v = -30$, speed $= 30\ \text{m s}^{-1}$ (g) $a = -10\ \text{m s}^{-2}$

5 (a) $a = -5e^{-t}$. (b) $e^{-t} > 0$ for all t so a < 0 for all $t \ge 0$

(c) $v = 5e^{-t} > 0$ for all $t \ge 0$ as $e^{-t} > 0$ for all t. Hence $\dfrac{dx}{dt}$ is never zero so x cannot have any stationary points.

6 (a) $20.61\ \text{km h}^{-1}$

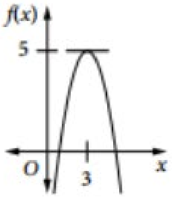
(b) Maximum velocity after 2.5 hours

CHAPTER REVIEW 14

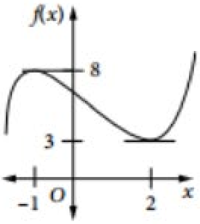
1 (a) $-0.5 < x < 2.5$ (b) $x < -0.5, x > 2.5$ (c) $x = 2.5$ (d) $x = -0.5$

2 (a) $x = -2, 1$ (b) $x < -2, x > 1$ (c) $-2 < x < 1$

3 (a)



(b)



4 (a) $(0, 0)$ max; $\left(\dfrac{4}{9}, \dfrac{-32}{243}\right)$ min (b) $(-1, 5)$ max; $(3, -27)$ min

5 $x = \dfrac{15\left(6 + \sqrt{3}\right)}{11}$, $y = \dfrac{45\left(5 - \sqrt{3}\right)}{22}$

6 $3\frac{13}{17}$ m, $4\frac{4}{17}$ m

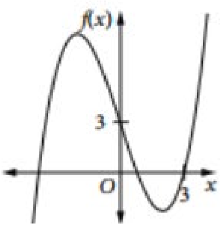
7 height $= y$ cm; $2x + 2y + \pi x = 50$, $y = \dfrac{50 - (\pi + 2)x}{2}$;

$\dfrac{100}{\pi + 4}$ by $\dfrac{50}{\pi + 4}$, $\dfrac{1250}{\pi + 4}\ \text{cm}^2$

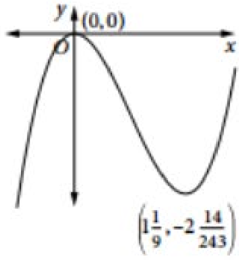
8 (a) $x = \pm 2$ (b) $-2 < x < 2$

(c) $8\frac{1}{3}$, local max; $-2\frac{1}{3}$, local min

(d)



9

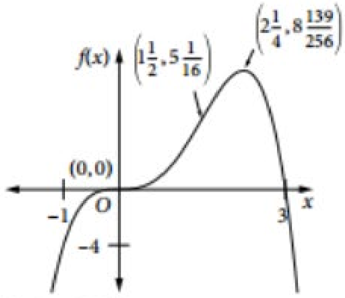


10 $\dfrac{5 - \sqrt{7}}{3}$ cm

11 $y' = 6x - 3x^2 = 3x(2 - x)$

At $x = 0$, $y = 0$, $y' = 0$, so the curve is horizontal where it cuts the y-axis at the origin and therefore crosses the y-axis at right angles.

12



$\left(2\frac{1}{4}, 8\frac{139}{256}\right)$ maximum turning point;

$(0, 0)$ horizontal inflection, $\left(1\frac{1}{2}, 5\frac{1}{16}\right)$ inflection

13 (a) (i) $\dfrac{\sqrt{9 + x^2}}{10\sqrt{2}}$ hours (ii) $\dfrac{4 - x}{20}$ hours

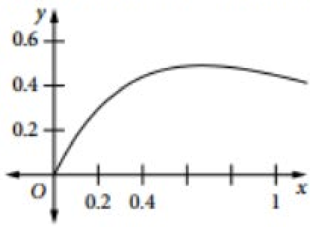
(b) $\dfrac{\sqrt{9 + x^2}}{10\sqrt{2}} + \dfrac{4 - x}{20}$

(c) $x = 3$ km, $t = 0.35$ h $= 21$ min

14 $60\ \text{km h}^{-1}$

15 (a) 0.4905 at $x = \frac{2}{3}$

(b) $f(0) = 0$, $f(0.5) = 0.472$, $f(1) = 0.446$



16 $\dfrac{d}{dt}(\theta_0 e^{-kt}) = -k\theta_0 e^{-kt} = -k\theta$

17 (a) \$10 000 (b) \$3012

(c) (i) \$602.39 per year (ii) \$1000 per year

(d) 11.5 years

18 $\dfrac{dy}{dx} = 6 \sec 2x \tan 2x$. Stationary points: $x = 0, \dfrac{\pi}{2}$

$\dfrac{d^2y}{dx^2} = 12 \sec 2x(\tan^2 2x + \sec^2 2x)$

Minimum at $(0, 3)$, maximum at $\left(\dfrac{\pi}{2}, -3\right)$.