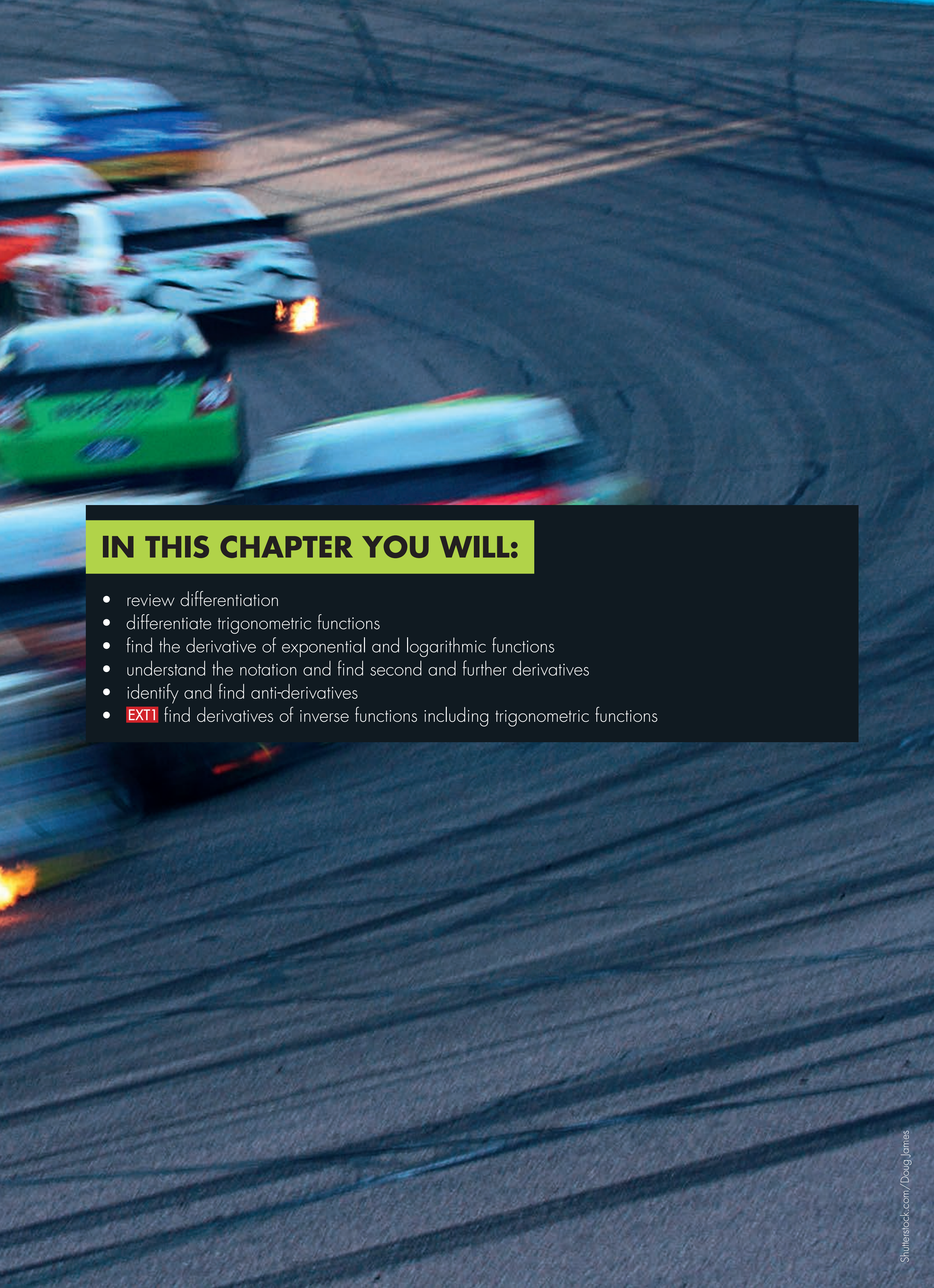
CALCULUS

5. FURTHER DIFFERENTIATION

In this chapter, you will review differentiation and learn how to differentiate trigonometric, exponential and logarithmic functions, and inverse functions, including inverse trigonometric functions. You will also look at higher derivatives and anti-derivatives.

CHAPTER OUTLINE

5.01 Differentiation review
5.02 Derivative of exponential functions
5.03 Derivative of logarithmic functions
5.04 Derivative of trigonometric functions
5.05 Second derivatives
5.06 Anti-derivative graphs
5.07 Anti-derivatives
5.08 Further anti-derivatives
5.09 EXT1 Derivative of inverse functions
5.10 EXT1 Derivative of inverse trigonometric functions



IN THIS CHAPTER YOU WILL:

- review differentiation
- differentiate trigonometric functions
- find the derivative of exponential and logarithmic functions
- understand the notation and find second and further derivatives
- identify and find anti-derivatives
- EXT1 find derivatives of inverse functions including trigonometric functions

TERMINOLOGY

anti-derivative: A function $F(x)$ whose derivative is $f(x)$, that is, $F'(x) = f(x)$. Also called the **primitive** or **integral** function.

anti-differentiation: The process of finding the original function given its derivative.

second derivative: The derivative $f''(x)$ or $\frac{d^2y}{dx^2}$; the derivative of the derivative $f'(x)$ or $\frac{dy}{dx}$.

5.01 Differentiation review

Chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} \qquad \frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$$

Product rule

If $y = uv$, then $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ or $y' = u'v + v'u$.

Quotient rule

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ or $y' = \frac{u'v - v'u}{v^2}$.

Rates of change

The **average rate of change** between 2 points (x_1, y_1) and (x_2, y_2) is the gradient:

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The **instantaneous rate of change** at point (x, y) is the derivative $f'(x)$ or $\frac{dy}{dx}$.

EXAMPLE 1

Water is pumped into a dam according to the formula $Q = 3t^3 + 2t^2 + 270$ where Q is the amount of water in kL and t is time in hours. Find:

a the amount of water in the dam after 6 hours

b the average rate at which the water is pumped into the dam between 3 and 6 hours

c the rate of change after 6 hours

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Solution

a $Q = 3t^3 + 2t^2 + 270$

When $t = 6$

$Q = 3(6)^3 + 2(6)^2 + 270$

$= 990$

So there is 990 kL of water in the dam after 6 hours.

b When $t = 3$

$Q = 3(3)^3 + 2(3)^2 + 270$

$= 369$

$$\text{Average rate of change} = \frac{Q_2 - Q_1}{t_2 - t_1} = \frac{990 - 369}{6 - 3} = \frac{621}{3} = 207$$

So average rate of change is 207 kL h^{-1}.

c $\frac{dQ}{dt} = 9t^2 + 4t$

When $t = 6$

$\frac{dQ}{dt} = 9(6)^2 + 4(6)$

$= 348$

So the rate of increase after 6 hours is 348 kL h^{-1}.

Exercise 5.01 Differentiation review

1 Differentiate each function.

a $3x^4 - 2x^3 + 7x - 4$ **b** $2x + 5$ **c** $6x^2 - 3x - 2$

2 Find the derivative $f'(x)$ given $f(x) = 4x^5 + 9x^2$.

3 Find $\frac{dx}{dt}$ if $x = 2\pi t^3 - 3t^2 + 1$.

4 Find $f'(-2)$ when $f(x) = 8x^3 + 5x - 2$.

5 Differentiate:

a x^{-5} **b** $x^{\frac{2}{3}}$ **c** $\frac{1}{x^2}$

d $\sqrt[4]{x}$ **e** $-\frac{5}{x^4}$

6 Find the derivative of $y = \sqrt[3]{x}$ at the point where $x = 8$.

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7 Differentiate:

a $(3x-1)^7$ **b** $(x^2-x+2)^3$ **c** $\sqrt{7x-2}$

d $\frac{1}{3x-2}$ **e** $\sqrt[3]{x^2-3}$

8 Find the derivative of:

a $x^2(x+4)$ **b** $(2x-1)(6x+5)$ **c** $4x(x^2+1)$

d $(4x+3)(x^2-1)^2$ **e** $2x^3\sqrt{x+1}$

9 Differentiate:

a $\frac{2x+3}{x-5}$ **b** $\frac{x^3}{4x-7}$ **c** $\frac{x^2+3}{2x-3}$

d $\frac{3x+1}{(2x+9)^2}$ **e** $\frac{3x+4}{\sqrt{2x-1}}$

10 Find the gradient of the tangent to the curve:

a $y = x^2 - 2x + 5$ at the point where $x = -2$

b $f(x) = x^3 - 3$ at the point $(-1, -4)$

11 Find the gradient of the normal to the curve:

a $f(x) = 3x^4 + x^2 - 2$ at the point where $x = -1$

b $y = x^2 + x - 3$ at the point $(-3, 3)$

12 Find the equation of the tangent to the curve:

a $y = 2x^2 - 5x - 6$ at the point $(3, -3)$

b $y = 5x^3 - 2x^2 - x$ at the point where $x = 2$

13 Find the equation of the normal to the curve:

a $f(x) = x^3 + 2x^2 - 3x - 5$ at the point $(-1, -1)$

b $y = x^2 - 3x + 1$ at the point where $x = 3$

14 For the curve $y = x^2 - 8x + 15$, find any values of x for which $\frac{dy}{dx} = 0$.

15 Find the coordinates of the points at which the curve $y = x^3 - 2$ has a tangent with gradient 12.

16 Function $f(x) = x^2 + x - 4$ has a tangent parallel to the line $3x + y - 4 = 0$ at point P. Find the equation of the tangent at P.

17 Find the coordinates of P if the gradient of the tangent to $y = \sqrt{x}$ is $\frac{1}{4}$ at point P.

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18 For the curve $y = \dfrac{5x-3}{4x+1}$ at the point where $x = 0$, find the equation of:

a the tangent **b** the normal

19 Find a formula for the rate of change $\dfrac{dQ}{dt}$ given:

a $Q = 3t^2 + 8$ **b** $Q = \dfrac{2}{t-3}$ **c** $Q = \sqrt[3]{2x+3}$

20 The mass M in kg of a snowball as it rolls down a hill over time t seconds is given by $M = t^2 + 3t + 4$.

a Find the average rate at which the mass changes between:

i 2 and 5 seconds **ii** 6 and 8 seconds

b Find the rate at which the mass is changing after:

i 5 seconds **ii** a minute

21 According to Boyle's Law, the pressure of a gas in pascals (Pa) is given by the formula $P = \dfrac{k}{V}$, where k is a constant and V is the volume of the gas in m^3. If $k = 250$ for a certain gas, find the rate of change in the pressure when $V = 10.7$.

22 The height of a ball in metres is given by $h = 4t - 2t^2$ where t is time in seconds.

a Find the height after:

i 1 s **ii** 1.5 s

b How long does it take for the ball to reach the ground?

c Find the velocity of the ball after:

i 0.5 s **ii** 1 s **iii** 2 s

5.02 Derivative of exponential functions

WS Derivatives of exponential functions

You learned how to differentiate $y = e^x$ in Year 11, in Chapter 10, *Exponential and logarithmic functions*.

Differentiation rules for e^x

$$\frac{d}{dx}e^x = e^x$$

$$\text{If } y = e^{f(x)} \text{ then } \frac{dy}{dx} = f'(x)\, e^{f(x)}$$

EXAMPLE 2

a If $f(x) = 3e^x$, find the equation of the tangent to the curve at $(2, 3e^2)$.

b Differentiate :

i x^2e^x **ii** e^{8x} **iii** e^{5x-2}

Solution

a
$$f(x) = 3e^x$$
$$f'(x) = 3e^x$$
At $(2, 3e^2)$
$$f'(2) = 3e^2$$
So $m = 3e^2$

Equation:
$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3e^2 &= 3e^2(x - 2) \\ &= 3e^2x - 6e^2 \\ y &= 3e^2x - 3e^2 \end{aligned}$$
(or $3e^2x - y - 3e^2 = 0$)

b **i**
$$y' = u'v + v'u$$
where $u = x^2$ and $v = e^x$
$$u' = 2x \qquad v' = e^x$$
$$\begin{aligned} y' &= 2xe^x + e^xx^2 \\ &= xe^x(2 + x) \end{aligned}$$

ii
$$\begin{aligned} \frac{dy}{dx} &= ae^{ax} \\ &= 8e^{8x} \end{aligned}$$

iii
$$\begin{aligned} \frac{dy}{dx} &= f'(x)e^{f(x)} \\ &= 5e^{5x-2} \end{aligned}$$

We can differentiate other exponential functions.

EXAMPLE 3

Differentiate 2^x.

Solution

$$2 = e^{\ln 2}$$
$$\begin{aligned} 2^x &= (e^{\ln 2})^x \\ &= e^{x \ln 2} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \ln 2 \; e^{x \ln 2} \\ &= \ln 2 \times 2^x \\ &= 2^x \ln 2 \end{aligned}$$

ln 2 is a constant

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Derivative of a^x

$$\text{If } y = a^x \text{, then } \frac{dy}{dx} = a^x \ln a$$

The proof of this has the same steps as in the previous example.

Exercise 5.02 Derivative of exponential functions

1 Differentiate:

a e^{7x} **b** e^{-x} **c** e^{6x-2} **d** e^{x^2+1}

e e^{x^3+5x+7} **f** e^{5x} **g** e^{-2x} **h** e^{10x}

i $e^{2x} + x$ **j** $x^2 + 2x + e^{1-x}$ **k** $(x + e^{4x})^5$ **l** xe^{2x}

m $\dfrac{e^{3x}}{x^2}$ **n** x^3e^{5x} **o** $\dfrac{e^{2x+1}}{2x+5}$

2 If $f(x) = e^{3x-2}$ find the exact value of $f'(1)$.

3 Find the derivative of:

a 3^x **b** 10^x **c** 2^{3x-4}

4 Find the gradient of the tangent to the curve $y = e^{5x}$ at the point where $x = 0$.

5 Find the equation of the tangent to the curve $y = e^{2x} - 3x$ at the point $(0, 1)$.

6 For the curve $y = e^{3x}$ at the point where $x = 1$, find the exact gradient of:

a the tangent **b** the normal

7 For the curve $y = e^{x^2}$ at the point $(1, e)$, find the equation of:

a the tangent **b** the normal

8 Find the equation of the tangent to the curve $y = 4^{x+1}$ at the point $(0, 4)$.

9 The population of a city is given by $P = 24\,500e^{0.038t}$ where t is time in years.

a Find the population after:

i 5 years **ii** 10 years

b Find the average rate of change in population between:

i the 1st and 5th years **ii** the 5th and 10th years

c Find the rate of change in population after:

i 5 years **ii** 10 years

10 The displacement of a particle is given by $s = 10e^{2t} - 5t$ cm after t minutes.

a Find the average rate of change in displacement between 1 and 5 minutes.

b Find the rate of change in displacement after:

i 1 minute **ii** 2 minutes **iii** 8 minutes

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11 A radioactive substance has a mass of $M = 20e^{-0.021t}$ in grams over time t years.

- **a** Find the initial mass.
- **b** Find the mass after 50 years.
- **c** Find the average rate of change in mass between 50 and 100 years.
- **d** Find the rate of change in mass after:
  - **i** 50 years **ii** 100 years **iii** 200 years

12 An object moves according to the formula $x = 3e^{2t}$ where x is displacement in cm and t is time in s.

- **a** Find the displacement at 5 s.
- **b** Find the velocity at 5 s.

INVESTIGATION

DERIVATIVE OF A LOGARITHMIC FUNCTION

Draw the derivative (gradient) function of a logarithm function.

What is the shape of the derivative function?

WS Derivatives of logarithmic functions

WS Exponential and logarithmic functions

5.03 Derivative of logarithmic functions

Logarithm rules

If $y = a^x$ then $\log_a y = x$

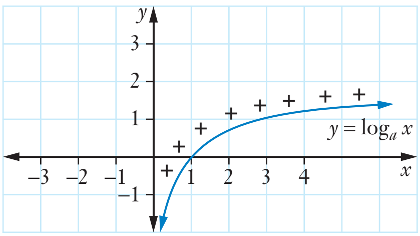
$$\log_a xy = \log_a x + \log_a y$$

$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

$$\log_a x^n = n \log_a x$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

To find the derivative of a logarithmic function, notice that the gradient of the function is always positive but is decreasing.



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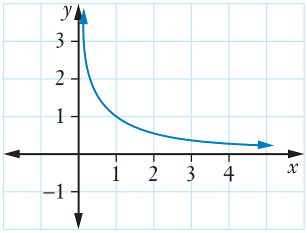
The derivative function of a logarithmic function is a hyperbola.



There is a special rule for $y = \ln x$.

Derivative of $y = \ln x$

$$\text{If } y = \ln x \text{, then } \frac{dy}{dx} = \frac{1}{x} \text{ where } x > 0.$$

Proof

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

Given $y = \ln x = \log_e x$

Then $x = e^y$

$$\frac{dx}{dy} = e^y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y} = \frac{1}{x}$$

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EXAMPLE 4

a Differentiate $(\ln x + 1)^3$.

b Find the equation of the tangent to the curve $y = \ln x$ at the point $(3, \ln 3)$.

Solution

a $(\ln x + 1)^3$ is a composite function in the form $y = [f(x)]^n$.

$$\begin{aligned}\frac{dy}{dx} &= f'(x)\, nf(x)^{n-1}\\ &= \frac{1}{x} \times 3(\ln x + 1)^2\\ &= \frac{3(\ln x + 1)^2}{x}\end{aligned}$$

b $\frac{dy}{dx} = \frac{1}{x}$

At $(3, \ln 3)$

$$\frac{dy}{dx} = \frac{1}{3}$$

So $m = \frac{1}{3}$

Equation:

$$\begin{aligned}y - y_1 &= m(x - x_1)\\ y - \ln 3 &= \frac{1}{3}(x - 3)\\ 3y - 3\ln 3 &= x - 3\\ 0 &= x - 3y - 3 + 3\ln 3\end{aligned}$$

Chain rule

If $y = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$ where $f(x) > 0$

Proof

$y = \ln f(x)$ is a composite function.

Let $y = \ln u$ and $u = f(x)$

$$\frac{dy}{du} = \frac{1}{u} \quad \text{and} \quad \frac{du}{dx} = f'(x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= \frac{1}{u} \times f'(x)\\ &= \frac{1}{f(x)} \times f'(x)\\ &= \frac{f'(x)}{f(x)}\end{aligned}$$

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EXAMPLE 5

a Differentiate:

i $\ln(x^2 - 3x + 1)$ **ii** $\ln\left(\frac{x+1}{3x-4}\right)$

b Find the gradient of the normal to the curve $y = \ln(x^3 - 5)$ at the point where $x = 2$.

Solution

a **i**
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$
$$= \frac{2x-3}{x^2-3x+1}$$

ii It is easier to simplify first using log laws.

$$y = \ln\left(\frac{x+1}{3x-4}\right)$$
$$= \ln(x+1) - \ln(3x-4)$$
$$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$$
$$= \frac{1}{x+1} - \frac{3}{3x-4}$$
$$= \frac{1(3x-4)}{(x+1)(3x-4)} - \frac{3(x+1)}{(3x-4)(x+1)}$$
$$= \frac{3x-4-3(x+1)}{(x+1)(3x-4)}$$
$$= \frac{3x-4-3x-3}{(x+1)(3x-4)}$$
$$= \frac{-7}{(x+1)(3x-4)}$$

b
$$\frac{dy}{dx} = \frac{3x^2}{x^3-5}$$

When $x = 2$

$$\frac{dy}{dx} = \frac{3(2)^2}{2^3-5}$$
$$= 4$$
$$m_1 = 4$$

The normal is perpendicular to the tangent:

$$m_1 m_2 = -1$$
$$4m_2 = -1$$
$$m_2 = -\frac{1}{4}$$

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We can differentiate logarithmic functions with a different base, a.

EXAMPLE 6

Differentiate $y = \log_2 x$.

Solution

$$y = \log_2 x$$
$$= \frac{\ln x}{\ln 2} \text{ using the change of base law}$$
$$= \frac{1}{\ln 2} \ln x$$

$$\frac{dy}{dx} = \frac{1}{\ln 2} \times \frac{1}{x}$$
$$= \frac{1}{x \ln 2}$$

ln 2 is a constant

Derivative of $\log_a x$

$$\text{If } y = \log_a x, \text{ then } \frac{dy}{dx} = \frac{1}{x \ln a}$$

The proof of this has the same steps as in the above example.

Exercise 5.03 Derivative of logarithmic functions

1 Differentiate:

a	$x + \ln x$	**b**	$1 - \ln 3x$	**c**	$\ln(3x + 1)$
d	$\ln(x^2 - 4)$	**e**	$\ln(5x^3 + 3x - 9)$	**f**	$\ln(5x + 1) + x^2$
g	$3x^2 + 5x - 5 + \ln 4x$	**h**	$\ln(8x - 9) + 2$	**i**	$\ln(2x + 4)(3x - 1)$
j	$\ln\left(\frac{4x+1}{2x-7}\right)$	**k**	$(1 + \ln x)^5$	**l**	$(\ln x - x)^9$
m	$(\ln x)^4$	**n**	$(x^2 + \ln x)^6$	**o**	$x \ln x$
p	$\frac{\ln x}{x}$	**q**	$(2x + 1)\ln x$	**r**	$x^3 \ln(x + 1)$
s	$\ln(\ln x)$	**t**	$\frac{\ln x}{x-2}$	**u**	$\frac{e^{2x}}{\ln x}$
v	$e^x \ln x$	**w**	$5(\ln x)^2$		

2 Find $f'(1)$ if $f(x) = \ln \sqrt{2 - x}$.

3 Find the derivative of $\log_{10} x$.

4 Find the equation of the tangent to the curve $y = \ln x$ at the point $(2, \ln 2)$.

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5 Find the equation of the tangent to the curve $y = \ln(x - 1)$ at the point where $x = 2$.

6 Find the gradient of the normal to the curve $y = \ln(x^4 + x)$ at the point $(1, \ln 2)$.

7 Find the exact equation of the normal to the curve $y = \ln x$ at the point where $x = 5$.

8 Find the equation of the tangent to the curve $y = \ln(5x + 4)$ at the point where $x = 3$.

9 Find the derivative of $\log_3(2x + 5)$.

10 Find the equation of the normal to the curve $y = \log_2 x$ at the point where $x = 2$.

11 The formula for the time t in years for kangaroo population growth on Kangaroo Island is given by $t = \dfrac{\ln\left(\dfrac{P}{20\,000}\right)}{0.021}$.

- **a** What is the initial population?
- **b** Find correct to one decimal place the time it takes for the population to grow to:
  - **i** 25 000
  - **ii** 50 000
- **c** Change the subject of the equation to P.
- **d** Find correct to the nearest whole number the average rate of change in population between 2 and 5 years.
- **e** Find correct to the nearest whole number the rate at which the population is growing after:
  - **i** 3 years
  - **ii** 5 years
  - **iii** 10 years

CLASS INVESTIGATION

DERIVATIVE OF TRIGONOMETRIC FUNCTIONS

1 Draw the derivative (gradient) function of sine, cosine and tangent functions.

What is the shape of the derivative function of each graph?

2 By substituting values of x in radians close to 0, find approximations to $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$, $\lim\limits_{x \to 0} \dfrac{\tan x}{x}$ and $\lim\limits_{x \to 0} \dfrac{\cos x}{x}$.

3 Differentiate by first principles to find the derivative of each trigonometric function using the above limits. The sine function will use the EXT1 trigonometric identity $\sin(A + B) = \sin A \cos B + \cos A \sin B$ from Chapter 4, *Trigonometric functions*.

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WS Derivatives of trigonometric functions

WS Trigonometric functions and gradient

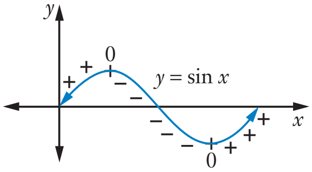
WS Further trigonometric equations

WS Differentiating trigonometric functions

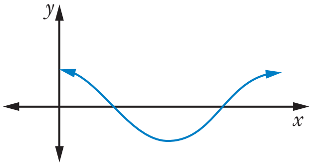
5.04 Derivative of trigonometric functions

Derivative of sin x

We can sketch the derivative (gradient) function of $y = \sin x$.



The sketch of the gradient function is $y = \cos x$.



Derivative of sin x

If $y = \sin x$, then $\dfrac{dy}{dx} = \cos x$

Proof

This proof uses trigonometric results from the investigation on the previous page.

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \times 0 + \cos x \times 1 \\
&= \cos x
\end{aligned}
$$

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EXAMPLE 7

a Differentiate $y = x \sin x$.

b Find the equation of the tangent to the curve $y = \sin x$ at the point $(\pi, 0)$.

Solution

a $y = x \sin x$ is in the form $y = uv$

where $u = x$ and $v = \sin x$

$u' = 1$ and $v' = \cos x$

$$\begin{aligned} y' &= u'v + v'u \\ &= 1 \times \sin x + \cos x \times x \\ &= \sin x + x \cos x \end{aligned}$$

b $\dfrac{dy}{dx} = \cos x$

At $(\pi, 0)$

$$\begin{aligned} \frac{dy}{dx} &= \cos \pi \\ &= -1 \end{aligned}$$

So $m = -1$

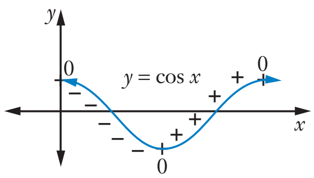
Equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= -1(x - \pi) \\ y &= -x + \pi \end{aligned}$$

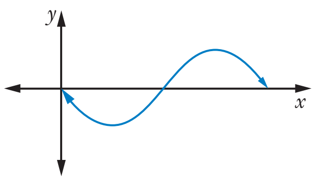
or $x + y - \pi = 0$

Derivative of cos x

We can sketch the derivative (gradient) function of $y = \cos x$.



The sketch of the gradient function below is $y = -\sin x$.



Derivative of cos x

$$\text{If } y = \cos x, \text{ then } \frac{dy}{dx} = -\sin x$$

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You can prove this in a similar way to the derivative of $y = \sin x$. A simpler proof involves changing $\cos x$ into $\sin\left(\frac{\pi}{2} - x\right)$ and using the derivative of $y = \sin x$.

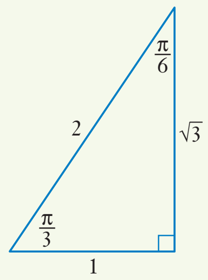
EXAMPLE 8

a Find the derivative of $y = \cos x$ at the point where $x = \frac{\pi}{3}$.

b Find the equation of the tangent to $y = \cos x$ at this point.

Solution

a $\frac{dy}{dx} = -\sin x$



When $x = \frac{\pi}{3}$

$$\frac{dy}{dx} = -\sin\frac{\pi}{3}$$

$$= -\frac{\sqrt{3}}{2}$$

b When $x = \frac{\pi}{3}$

$$y = \cos\frac{\pi}{3}$$

$$= \frac{1}{2}$$

Equation:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{1}{2} = -\frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right)$$

$$2y - 1 = -\sqrt{3}\left(x - \frac{\pi}{3}\right) \quad \text{(multiplying both sides by 2)}$$

$$= -\sqrt{3}x + \frac{\pi\sqrt{3}}{3}$$

$$6y - 3 = -3\sqrt{3}x + \pi\sqrt{3} \quad \text{(multiplying both sides by 3)}$$

$$3\sqrt{3}x + 6y - 3 - \pi\sqrt{3} = 0$$

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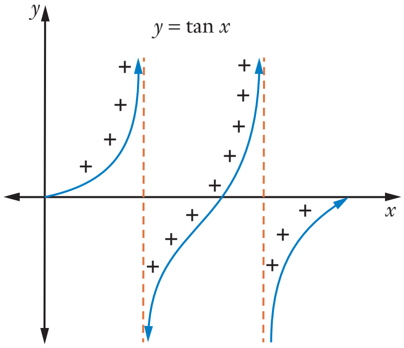
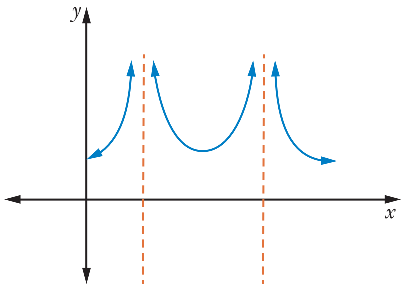
Derivative of tan x

We can sketch the derivative (gradient) function of $y = \tan x$. Notice that the gradient function will have asymptotes in the same place as the original graph, because this is where the tangent is vertical and the gradient is undefined.



The gradient function is $y = \sec^2 x$, where $\sec x = \dfrac{1}{\cos x}$.



Derivative of tan x

$$\text{If } y = \tan x, \text{ then } \frac{dy}{dx} = \sec^2 x$$

You can prove this in a similar way to the derivative of $y = \sin x$. A simpler proof involves changing $\tan x$ into $\dfrac{\sin x}{\cos x}$ and using the quotient rule.

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EXAMPLE 9

a Differentiate $y = \dfrac{\tan x}{3x^2}$.

b Find the gradient of the tangent to the curve $f(x) = \tan x$ at the point where $x = \dfrac{\pi}{4}$.

Solution

a $y = \dfrac{\tan x}{3x^2}$ is in the form $y = \dfrac{u}{v}$.

$u = \tan x$ and $v = 3x^2$

$u' = \sec^2 x$ $\quad v' = 6x$

$$y' = \frac{u'v - v'u}{u^2}$$

$$= \frac{\sec^2 x \times 3x^2 - 6x \times \tan x}{(3x^2)^2}$$

$$= \frac{3x(x\sec^2 x - 2\tan x)}{9x^4}$$

$$= \frac{x\sec^2 x - 2\tan x}{3x^3}$$

b $\dfrac{dy}{dx} = \sec^2 x$

At $x = \dfrac{\pi}{4}$

$$\frac{dy}{dx} = \sec^2 \frac{\pi}{4}$$

$$= (\sqrt{2})^2$$

$$= 2$$

Chain rule

If $y = \sin f(x)$, then $\dfrac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\dfrac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\dfrac{dy}{dx} = f'(x)\sec^2 f(x)$

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Here is the proof for $y = \sin f(x)$. The others are similar.

Proof

$y = \sin f(x)$ is a composite function

where $y = \sin u$ and $u = f(x)$

$$\frac{dy}{du} = \cos u \text{ and } \frac{du}{dx} = f'(x)$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos u \times f'(x) \\ &= f'(x) \cos u \\ &= f'(x) \cos f(x)\end{aligned}$$

EXAMPLE 10

a Differentiate each function.

i $y = \sin 7x$ **ii** $y = \cos\left(4x^3 + \frac{\pi}{3}\right)$ **iii** $y = \tan(5x - \pi)$

b Find the gradient of the normal to the curve $f(x) = \cos\frac{x}{2}$ at the point where $x = \pi$.

Solution

a **i** $$\begin{aligned}\frac{dy}{dx} &= f'(x) \cos f(x) \\ &= 7 \cos 7x\end{aligned}$$

ii $$\begin{aligned}\frac{dy}{dx} &= -f'(x) \sin f(x) \\ &= -12x^2 \sin\left(4x^3 + \frac{\pi}{3}\right)\end{aligned}$$

iii $$\begin{aligned}\frac{dy}{dx} &= f'(x) \sec^2 f(x) \\ &= 5 \sec^2 (5x - \pi)\end{aligned}$$

b $$\begin{aligned}\frac{dy}{dx} &= -f'(x) \sin f(x) \\ &= -\frac{1}{2} \sin\frac{x}{2}\end{aligned}$$

At $x = \pi$:

$$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{2} \sin\frac{\pi}{2} \\ &= -\frac{1}{2} \times 1 \\ &= -\frac{1}{2}\end{aligned}$$

So $m_1 = -\frac{1}{2}$

Normal is perpendicular to the tangent:

$$\begin{aligned}m_1 m_2 &= -1 \\ -\frac{1}{2} m_2 &= -1 \\ m_2 &= 2\end{aligned}$$

While trigonometric functions are usually expressed in radians, we can differentiate angles *in degrees* by using the conversion $\pi = 180°$.

EXAMPLE 11

Differentiate $y = \sin x°$.

Solution

$180° = \pi$ radians

$$1° = \frac{\pi}{180}$$

$$x° = \frac{\pi x}{180}$$

So $y = \sin x°$ becomes

$$y = \sin \frac{\pi x}{180}$$

$$\begin{aligned}\frac{dy}{dx} &= f'(x) \cos f(x) \\ &= \frac{\pi}{180} \cos \frac{\pi x}{180} \\ &= \frac{\pi}{180} \cos x°\end{aligned}$$

We can also differentiate composite functions involving trigonometric functions.

EXAMPLE 12

Differentiate:

a $\tan(e^x)$

b $\ln(\cos x)$

Solution

a
$$\begin{aligned}\frac{dy}{dx} &= f'(x) \sec^2 f(x) \\ &= e^x \sec^2(e^x)\end{aligned}$$

b
$$\begin{aligned}\frac{dy}{dx} &= \frac{f'(x)}{f(x)} \\ &= \frac{-\sin x}{\cos x} \\ &= -\tan x\end{aligned}$$

Exercise 5.04 Derivative of trigonometric functions

1 Differentiate:

a $\sin 4x$

b $\cos 3x$

c $\tan 5x$

d $\tan(3x + 1)$

e $\cos(-x)$

f $3 \sin x$

g $4 \cos(5x - 3)$

h $2 \cos(x^3)$

i $7 \tan(x^2 + 5)$

j $\sin 3x + \cos 8x$

k $\tan(\pi + x) + x^2$

l $x \tan x$

m $\sin 2x \tan 3x$

n $\dfrac{\sin x}{2x}$

o $\dfrac{3x + 4}{\sin 5x}$

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p $(2x + \tan 7x)^9$ **q** $\sin^2 x$ **r** $3\cos^3 5x$

s $e^x - \cos 2x$ **t** $\sin(1 - \ln x)$ **u** $\sin(e^x + x)$

v $\ln(\sin x)$ **w** $e^{3x} \cos 2x$ **x** $\dfrac{e^{2x}}{\tan 7x}$

2 Find the gradient of the tangent to the curve $y = \tan 3x$ at the point where $x = \dfrac{\pi}{9}$.

3 Find the equation of the tangent to the curve $y = \sin(\pi - x)$ at the point $\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$ in exact form.

4 Differentiate $\ln(\cos x)$.

5 Find the exact gradient of the normal to $y = \sin 3x$ at the point where $x = \dfrac{\pi}{18}$.

6 Differentiate $e^{\tan x}$.

7 Find the equation of the normal to the curve $y = 3\sin 2x$ at the point where $x = \dfrac{\pi}{8}$ in exact form.

8 Show that $\dfrac{d}{dx}[\ln(\tan x)] = \tan x + \cot x$.

9 Differentiate each function.

a $y = \tan x°$ **b** $y = 3\cos x°$ **c** $y = \dfrac{\sin x°}{5}$

10 Find the derivative of $\cos x \sin^4 x$.

11 The population of salmon in a salmon farm grows and reduces as fish are born and sold.

The population is given by $P = 225\cos\dfrac{2\pi t}{9} + 750$ where t is time in days.

a What is the centre of the population?

b What is the minimum number of salmon in the farm at any one time?

c What is the maximum population?

d At what times is the population 700?

e At what rate is the population changing after:

i 3 days? **ii** a week? **iii** 10 days? **iv** 18 days?

f At what times is the population growing at the rate of 25 fish per day?

12 The tide was measured over time at a beach at Merimbula and given the formula $D = 8\sin\dfrac{\pi t}{6} + 9$ where D is depth of water in metres and t is time in hours.

a How deep was the water:

i initially? **ii** after 5 hours?

b When was the water 10 m deep?

c At what rate was the depth changing after:

i 3 hours? **ii** 11 hours? **iii** 12 hours?

d At what times was the depth of water decreasing by 3 m h^{-1}?

WS
The second derivative

WS
First and second derivatives

5.05 Second derivatives

Second derivative

Differentiating $f(x)$ gives $f'(x)$, the first derivative.

Differentiating $f'(x)$ gives $f''(x)$, the **second derivative**.

It is also possible to differentiate further.

Using function notation, differentiating several times gives $f'(x), f''(x), f'''(x)$ and so on.

Using $\frac{dy}{dx}$ notation, differentiating several times gives $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ and so on.

The notation $\frac{d^2y}{dx^2}$ comes from $\frac{d^2}{dx^2}(y)$.

EXAMPLE 13

a Find the first 4 derivatives of $f(x) = x^3 - 4x^2 + 3x - 2$.

b Find the second derivative of $y = (2x + 5)^7$.

c If $f(x) = 4 \cos 3x$, show that $f''(x) = -9 f(x)$

Solution

a
$$f'(x) = 3x^2 - 8x + 3$$
$$f''(x) = 6x - 8$$
$$f'''(x) = 6$$
$$f''''(x) = 0$$

b
$$\begin{aligned}\frac{dy}{dx} &= f'(x) \times nf(x)^{n-1} \\ &= 2 \times 7(2x + 5)^6 \\ &= 14(2x + 5)^6\end{aligned}$$

$$\begin{aligned}\frac{d^2y}{dx^2} &= f'(x) \times nf(x)^{n-1} \\ &= 2 \times 6 \times 14(2x + 5)^5 \\ &= 168(2x + 5)^5\end{aligned}$$

c
$$\begin{aligned}f'(x) &= -f'(x) \times \sin f(x) \\ &= -3 \times 4 \sin 3x \\ &= -12 \sin 3x\end{aligned}$$

$$\begin{aligned}f''(x) &= f'(x) \times \cos f(x) \\ &= 3 \times (-12 \cos 3x) \\ &= -36 \cos 3x \\ &= -9(4 \cos 3x) \\ &= -9f(x) && \text{since } f(x) = 4 \cos 3x\end{aligned}$$

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Exercise 5.05 Second derivatives

1 Find the first 4 derivatives of $x^7 - 2x^5 + x^4 - x - 3$.

2 If $f(x) = x^9 - 5$, find $f''(x)$.

3 Find $f'(x)$ and $f''(x)$ if $f(x) = 2x^5 - x^3 + 1$.

4 Find $f'(1)$ and $f''(-2)$, given $f(t) = 3t^4 - 2t^3 + 5t - 4$.

5 Find the first 3 derivatives of $x^7 - 2x^6 + 4x^4 - 7$.

6 Find the first and second derivatives of $y = 2x^2 - 3x + 3$.

7 If $f(x) = x^4 - x^3 + 2x^2 - 5x - 1$, find $f'(-1)$ and $f''(2)$.

8 Find the first and second derivatives of x^{-4}.

9 If $g(x) = \sqrt{x}$, find $g''(4)$.

10 Given $h = 5t^3 - 2t^2 + t + 5$, find $\dfrac{d^2h}{dt^2}$ when $t = 1$.

11 Find any values of x for which $\dfrac{d^2y}{dx^2} = 3$, given $y = 3x^3 - 2x^2 + 5x$.

12 Find all values of x for which $f''(x) > 0$ given that $f(x) = x^3 - x^2 + x + 9$.

13 Find the first and second derivatives of $(4x - 3)^5$.

14 Find $f'(x)$ and $f''(x)$ if $f(x) = \sqrt{2 - x}$.

15 Find the first and second derivatives of $f(x) = \dfrac{x+5}{3x-1}$.

16 Find $\dfrac{d^2v}{dt^2}$ if $v = (t + 3)(2t - 1)^2$.

17 Find the value of b in $y = bx^3 - 2x^2 + 5x + 4$ if $\dfrac{d^2y}{dx^2} = -2$ when $x = \dfrac{1}{2}$.

18 Find $f''(1)$ if $f(t) = t(2t - 1)^7$.

19 Find the value of b if $f(x) = 5bx^2 - 4x^3$ and $f''(-1) = -3$.

20 If $y = e^{4x} + e^{-4x}$, show that $\dfrac{d^2y}{dx^2} = 16y$.

21 Prove that $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 0$ given $y = 3e^{2x}$.

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22 Show that $\frac{d^2y}{dx^2} = b^2y$ for $y = ae^{bx}$.

23 Find the value of n if $y = e^{3x}$ satisfies the equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + ny = 0$.

24 Show that $\frac{d^2y}{dx^2} = -25y$ if $y = 2 \cos 5x$.

25 Given $f(x) = -2 \sin x$, show that $f''(x) = -f(x)$.

26 If $y = 2 \sin 3x - 5 \cos 3x$, show that $\frac{d^2y}{dx^2} = -9y$.

27 Find values of a and b if $\frac{d^2y}{dx^2} = ae^{3x} \cos 4x + be^{3x} \sin 4x$, given $y = e^{3x} \cos 4x$.

28 Find the exact value of $f''(2)$ if $f(x) = x\sqrt{3x-4}$.

29 The displacement of a particle moving in a straight line is given by $x = 2t^3 - 5t^2 + 7t + 8$, where x is in metres and t is in seconds.

- **a** Find the initial displacement.
- **b** Find the displacement after 3 seconds.
- **c** Find the velocity after 3 seconds.
- **d** Find the acceleration after 3 seconds.

30 The height in cm of a pendulum as it swings is given by $h = 8 \cos \pi t + 12$ where t is time in seconds.

- **a** What is the height of the pendulum after 3 s?
- **b** What is the maximum and minimum height of the pendulum?
- **c** What is the velocity of the pendulum after:
  - **i** 1 s? **ii** 1.5 s?
- **d** What is the acceleration of the pendulum:
  - **i** initially? **ii** after 1 s? **iii** after 1.5 s?
- **e** EXT1 Write the equation for acceleration in terms of h.

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5.06 Anti-derivative graphs

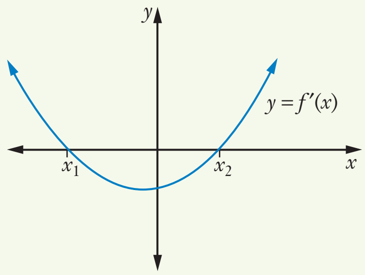
The process of finding the original function $y = f(x)$ given the derivative $y = f'(x)$ is called **anti-differentiation**, and the original function is called the **anti-derivative** function, also called the **primitive** or **integral function**.

WS Shapes of curves

Anti-derivative graphs

EXAMPLE 14

Sketch the graph of the anti-derivative (primitive function) given the graph of the derivative function below and an initial condition, or starting point, of (0, 2).



Solution

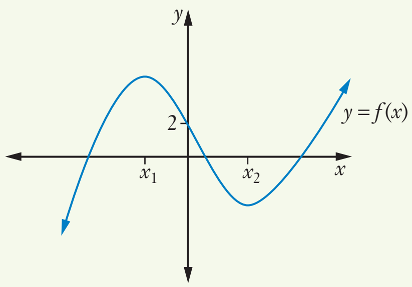
Remember that when you sketch a derivative function, the x-intercepts are where the original function has zero gradient, or stationary (turning) points.

On this graph the stationary points are at $x = x_1$ and $x = x_2$.

Above the x-axis shows where the original function has a positive gradient (it is increasing). On this graph, this is where $x < x_1$ and $x > x_2$.

Below the x-axis shows where the original function has a negative gradient (it is decreasing). On this graph, this is where $x_1 < x < x_2$.

We can sketch this information together with the point (0, 2):



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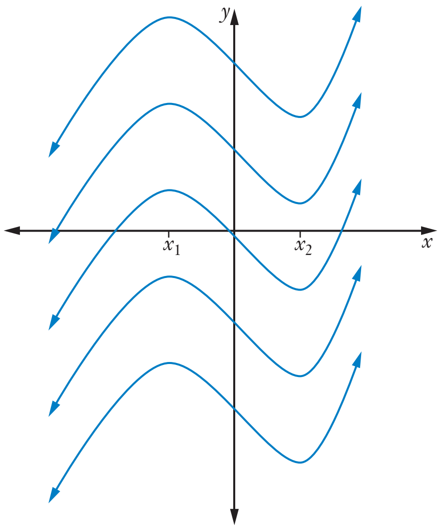
We are not given enough information to sketch a unique graph. There is no way of knowing what the y values of the stationary points are or the stretch or compression of the graph. Also, if we are not given a fixed point on the function, we could sketch many graphs that satisfy the information from the derivative function.

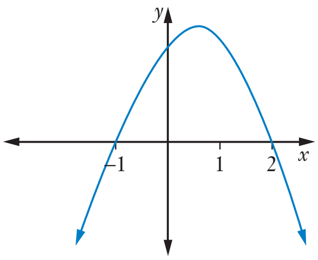


The anti-derivative gives a **family** of curves.

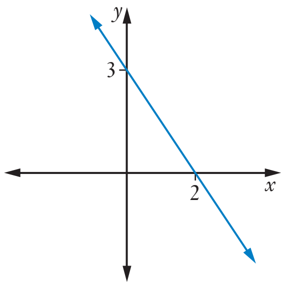
Exercise 5.06 Anti-derivative graphs

1 For each function graphed, sketch the graph of the anti-derivative function given it passes through:

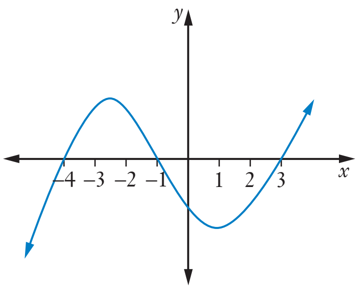
a (0, –1)



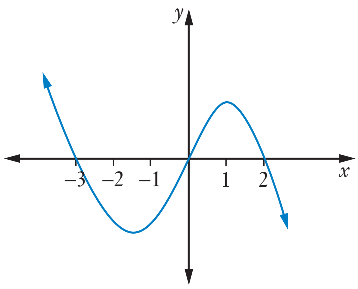
b (1, 2)



c (0, 3)



d (–1, –1)



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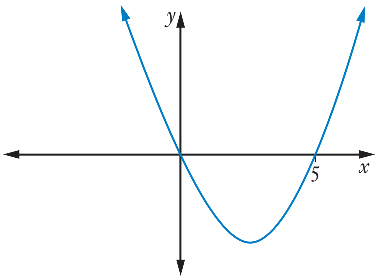
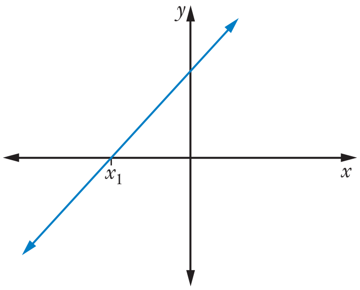
e (0, 1)

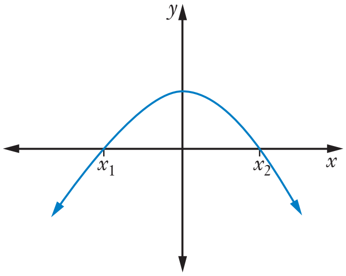


2 Sketch a family of graphs that could represent the anti-derivative function of each graph.

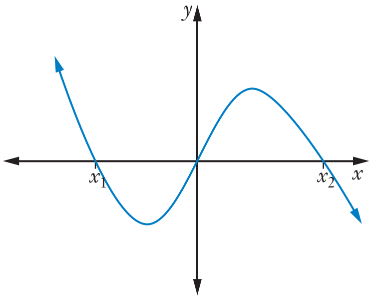
a



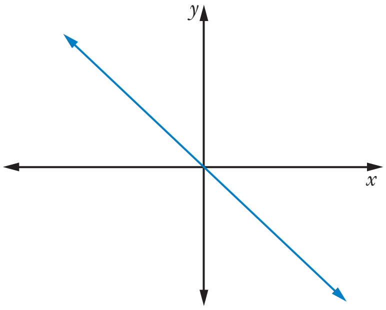
b



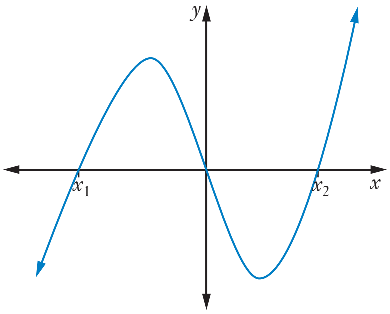
c



d



e



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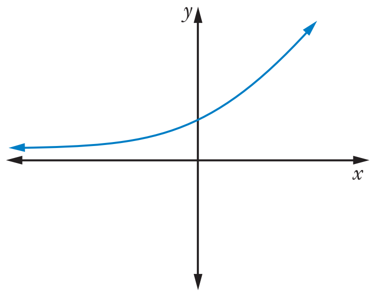
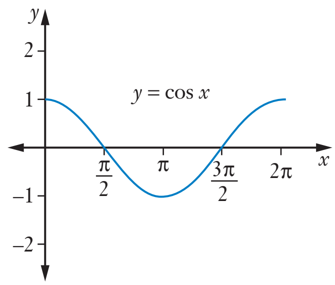
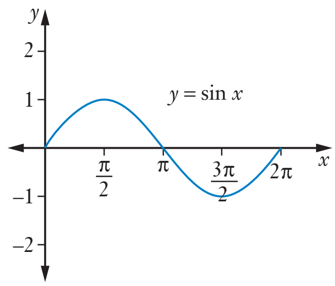
3 The anti-derivative function of the graph below passes through (0, –1). Sketch its graph.



4 Sketch the graph of the anti-derivative function of $y = \cos x$ given that it passes through (0, 0).



5 Sketch a family of anti-derivative functions for the graph below.



INVESTIGATION

ANTI-DERIVATIVE OF $y = x^n$

1 Differentiate:

a x^2 **b** $x^2 + 5$ **c** $x^2 - 7$ **d** $x^2 + 4$ **e** $x^2 - 75$

What would be the anti-derivative of $2x$?

2 Differentiate:

a x^3 **b** $x^3 + 1$ **c** $x^3 + 6$ **d** $x^3 - 2$ **e** $x^3 - 14$

What would be the anti-derivative of $3x^2$?

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3 Differentiate:

a x^4 **b** $x^4 - 3$ **c** $x^4 + 2$ **d** $x^4 + 10$ **e** $x^4 - 1$

What would be the anti-derivative of $4x^3$?

4 Differentiate:

a x^n **b** $x^n + 7$ **c** $x^n + 9$ **d** $x^n - 5$ **e** $x^n - 2$

What would be the anti-derivative of nx^{n-1}?

Can you find a general rule for anti-derivatives that would work for these examples?

5.07 Anti-derivatives

Since anti-differentiation is the reverse of differentiation, we can find the equation of an anti-derivative function.

WS Anti-derivatives

WS Antiderivatives

WS Antidifferentiation

Anti-derivative of x^n

If $\frac{dy}{dx} = x^n$, then $y = \frac{1}{n+1}x^{n+1} + C$ where C is a constant.

Proof

$$\frac{d}{dx}\left(\frac{1}{n+1}x^{n+1} + C\right) = \frac{(n+1)x^n}{n+1}$$
$$= x^n$$

We can apply the same rules to anti-derivatives as we use for derivatives. Here are some of the main ones we use.

Anti-derivative rules

If $\frac{dy}{dx} = k$ then $y = kx$.

If $\frac{dy}{dx} = kx^n$ then $y = \frac{1}{n+1}kx^{n+1} + C$.

If $\frac{dy}{dx} = f(x) + g(x)$ then $y = F(x) + G(x) + C$ where $F(x)$ and $G(x)$ are the anti-derivatives of $f(x)$ and $g(x)$ respectively.

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EXAMPLE 15

Find the anti-derivative of $x^4 - 4x^3 + 9x^2 - 6x + 5$.

Solution

If $f(x) = x^4 - 4x^3 + 9x^2 - 6x + 5$

$$F(x) = \frac{1}{5}x^5 - 4 \times \frac{1}{4}x^4 + 9 \times \frac{1}{3}x^3 - 6 \times \frac{1}{2}x^2 + 5x + C$$

Anti-derivative of 5 is 5*x*.

$$= \frac{x^5}{5} - x^4 + 3x^3 - 3x^2 + 5x + C$$

If we have some information about the anti-derivative function, we can use this to evaluate the constant C.

The anti-derivative function

EXAMPLE 16

a The gradient of a curve is given by $\frac{dy}{dx} = 6x^2 + 8x$. If the curve passes through the point $(1, -3)$, find its equation.

b If $f''(x) = 6x + 2$ and $f'(1) = f(-2) = 0$, find $f(3)$.

Solution

a $\frac{dy}{dx} = 6x^2 + 8x$

So $y = 6 \times \frac{1}{3}x^3 + 8 \times \frac{1}{2}x^2 + C$

$= 2x^3 + 4x^2 + C$

Substitute $(1, -3)$:

$-3 = 2(1)^3 + 4(1)^2 + C$

$= 6 + C$

$-9 = C$

Equation is $y = 2x^3 + 4x^2 - 9$.

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b $f''(x) = 6x + 2$

$$f'(x) = 6 \times \frac{1}{2}x^2 + 2 \times \frac{1}{1}x^1 + C$$
$$= 3x^2 + 2x + C$$

Since $f'(1) = 0$:

$$0 = 3(1)^2 + 2(1) + C$$
$$= 5 + C$$
$$-5 = C$$

So $f'(x) = 3x^2 + 2x - 5$

$$f(x) = 3 \times \frac{1}{3}x^3 + 2 \times \frac{1}{2}x^2 - 5 \times \frac{1}{1}x^1 + D$$
$$= x^3 + x^2 - 5x + D$$

Since $f(-2) = 0$:

$$0 = (-2)^3 + (-2)^2 - 5(-2) + D$$
$$= -8 + 4 + 10 + D$$
$$= 6 + D$$
$$-6 = D$$

Equation is $f(x) = x^3 + x^2 - 5x - 6$

$$f(3) = 3^3 + 3^2 - 5(3) - 6$$
$$= 27 + 9 - 15 - 6$$
$$= 15$$

Chain rule

If $\frac{dy}{dx} = (ax + b)^n$, then $y = \frac{1}{a(n+1)}(ax + b)^{n+1} + C$ where C is a constant, $a \neq 0$ and $n \neq -1$.

Proof

$$\frac{d}{dx}\left(\frac{1}{a(n+1)}(ax+b)^{n+1} + C\right) = \frac{a(n+1)(ax+b)^n}{a(n+1)}$$
$$= (ax + b)^n$$

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EXAMPLE 17

a Find the anti-derivative of $(3x + 7)^8$.

b The gradient of a curve is given by $\frac{dy}{dx} = (2x - 3)^4$. If the curve passes through the point $(2, -7)$, find its equation.

Solution

a
$$\frac{dy}{dx} = (3x + 7)^8$$
$$\begin{aligned} y &= \frac{1}{a(n+1)}(ax + b)^{n+1} + C \\ &= \frac{1}{3(8+1)}(3x + 7)^{8+1} + C \\ &= \frac{1}{27}(3x + 7)^9 + C \\ &= \frac{(3x+7)^9}{27} + C \end{aligned}$$

b
$$\frac{dy}{dx} = (2x - 3)^4$$
$$\begin{aligned} y &= \frac{1}{a(n+1)}(ax + b)^{n+1} + C \\ &= \frac{1}{2(4+1)}(2x - 3)^{4+1} + C \\ &= \frac{1}{10}(2x - 3)^5 + C \end{aligned}$$

Substitute $(2, -7)$:
$$\begin{aligned} -7 &= \frac{1}{10}(2 \times 2 - 3)^5 + C \\ &= \frac{1}{10}(1)^5 + C \\ &= \frac{1}{10} + C \end{aligned}$$
$$-7\frac{1}{10} = C$$

So the equation is $y = \frac{1}{10}(2x - 3)^5 - 7\frac{1}{10}$
$$= \frac{(2x-3)^5 - 71}{10}$$

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General chain rule

If $\dfrac{dy}{dx} = f'(x)[f(x)]^n$ then $y = \dfrac{1}{n+1}[f(x)]^{n+1} + C$ where C is a constant and $n \neq -1$

Proof

$$\frac{d}{dx}\left(\frac{1}{n+1}[f(x)]^{n+1} + C\right) = \frac{1}{n+1}f'(x)(n+1)[f(x)]^{n+1-1}$$
$$= f'(x)[f(x)]^n$$

EXAMPLE 18

Find the anti-derivative of:

a $8x^3(2x^4 - 1)^5$

b $x^2(x^3 + 2)^7$

Solution

a Given $f(x) = 2x^4 - 1$

$$f'(x) = 8x^3$$
$$\frac{dy}{dx} = 8x^3(2x^4 - 1)^5$$
$$= f'(x)[f(x)]^n$$
$$y = \frac{1}{n+1}f(x)^{n+1} + C$$
$$= \frac{1}{5+1}(2x^4 - 1)^{5+1} + C$$
$$= \frac{1}{6}(2x^4 - 1)^6 + C$$
$$= \frac{(2x^4 - 1)^6}{6} + C$$

b Given $f(x) = x^3 + 2$

$$f'(x) = 3x^2$$
$$\frac{dy}{dx} = x^2(x^3 + 2)^7$$
$$= \frac{1}{3} \times 3x^2(x^3 + 2)^7$$
$$= \frac{1}{3}f'(x)[f(x)]^n$$
$$y = \frac{1}{3} \times \frac{1}{n+1}f(x)^{n+1} + C$$
$$= \frac{1}{3} \times \frac{1}{7+1}(x^3 + 2)^{7+1} + C$$
$$= \frac{1}{24}(x^3 + 2)^8 + C$$
$$= \frac{(x^3 + 2)^8}{24} + C$$

Exercise 5.07 Anti-derivatives

1 Find the anti-derivative of:

a $2x - 3$ **b** $x^2 + 8x + 1$ **c** $x^5 - 4x^3$

d $(x - 1)^2$ **e** 6 **f** $(3x + 2)^5$

g $8(2x - 7)^4$

2 Find $f(x)$ if:

a $f'(x) = 6x^2 - x$ **b** $f'(x) = x^4 - 3x^2 + 7$ **c** $f'(x) = x - 2$

d $f'(x) = (x + 1)(x - 3)$ **e** $f'(x) = x^{\frac{1}{2}}$

3 Express y in terms of x if:

a $\frac{dy}{dx} = 5x^4 - 9$ **b** $\frac{dy}{dx} = x^{-4} - 2x^{-2}$ **c** $\frac{dy}{dx} = \frac{x^3}{5} - x^2$

d $\frac{dy}{dx} = \frac{2}{x^2}$ **e** $\frac{dy}{dx} = x^3 - \frac{2x}{3} + 1$

4 Find the anti-derivative of:

a $\sqrt{x}$ **b** x^{-3} **c** $\frac{1}{x^8}$

d $x^{-\frac{1}{2}} + 2x^{-\frac{2}{3}}$ **e** $x^{-7} - 2x^{-2}$

5 Find the anti-derivative of:

a $2x(x^2 + 5)^4$ **b** $3x^2(x^3 - 1)^9$ **c** $8x(2x^2 + 3)^3$

d $15x^4(x^5 + 1)^6$ **e** $x(x^2 - 4)^7$ **f** $x^5(2x^6 - 7)^8$

g $(2x - 1)(x^2 - x + 3)^4$ **h** $(3x^2 + 4x - 7)(x^3 + 2x^2 - 7x)^{10}$

i $(x - 3)(x^2 - 6x - 1)^5$

6 If $\frac{dy}{dx} = x^3 - 3x^2 + 5$ and $y = 4$ when $x = 1$, find an equation for y in terms of x.

7 If $f'(x) = 4x - 7$ and $f(2) = 5$, find an equation for $y = f(x)$.

8 Given $f'(x) = 3x^2 + 4x - 2$ and $f(-3) = 4$, find the value of $f(1)$.

9 Given that the gradient of the tangent to a curve is given by $\frac{dy}{dx} = 2 - 6x$ and the curve passes through $(-2, 3)$, find the equation of the curve.

10 If $\frac{dx}{dt} = (t - 3)^2$ and $x = 7$ when $t = 0$, find x when $t = 4$.

11 Given $\frac{d^2y}{dx^2} = 8$, and $\frac{dy}{dx} = 0$ and $y = 3$ when $x = 1$, find the equation of y in terms of x.

12 If $\frac{d^2y}{dx^2} = 12x + 6$ and $\frac{dy}{dx} = 1$ at the point $(-1, -2)$, find the equation of the curve.

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13 If $f''(x) = 6x - 2$ and $f'(2) = f(2) = 7$, find the equation of the function $y = f(x)$.

14 Given $f''(x) = 5x^4$, $f'(0) = 3$ and $f(-1) = 1$, find $f(2)$.

15 A curve has $\dfrac{d^2y}{dx^2} = 8x$ and the tangent at $(-2, 5)$ has an angle of inclination of 45° with the x-axis. Find the equation of the curve.

16 The tangent to a curve with $\dfrac{d^2y}{dx^2} = 2x - 4$ makes an angle of inclination of 135° with the x-axis at the point $(2, -4)$. Find its equation.

17 A function has a tangent parallel to the line $4x - y - 2 = 0$ at the point $(0, -2)$, and $f''(x) = 12x^2 - 6x + 4$. Find the equation of the function.

18 A curve has $\dfrac{d^2y}{dx^2} = 6$ and the tangent at $(-1, 3)$ is perpendicular to the line $2x + 4y - 3 = 0$. Find the equation of the curve.

19 A function has $f'(1) = 3$ and $f(1) = 5$. Evaluate $f(-2)$ given $f''(x) = 6x + 18$.

20 The velocity of an object is given by $\dfrac{dx}{dt} = 6t - 5$. If the object has initial displacement of -2, find the equation for the displacement.

21 The acceleration of a particle is given by $\dfrac{d^2x}{dt^2} = 24t^2 - 12t + 6$ m s^{-2}. Its velocity $\dfrac{dx}{dt} = 0$ when $t = 1$ and its displacement $x = -3$ when $t = 0$. Find the equation for its displacement.

5.08 Further anti-derivatives

Anti-derivative of exponential functions

$$\text{If } \frac{dy}{dx} = e^x, \text{ then } y = e^x + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = e^{ax+b}, \text{ then } y = \frac{1}{a}e^{ax+b} + C$$

$$\text{If } \frac{dy}{dx} = f'(x)e^{f(x)}, \text{ then } y = e^{f(x)} + C$$

Proof (by differentiation)

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{a}e^{ax+b} + C\right) &= \frac{1}{a} \times ae^{ax+b} \\ &= e^{ax+b}\end{aligned}$$

$$\frac{d}{dx}[e^{f(x)} + C] = f'(x)e^{f(x)}$$

EXAMPLE 19

a Find the anti-derivative of $e^{4x} + 1$.

b Find the equation of the function $y = f(x)$ given $f'(x) = 6e^{3x}$ and $f(2) = 2e^6$.

Solution

a $\frac{1}{a}e^{ax+b} + C = \frac{1}{4}e^{4x} + C$

b
$$f'(x) = 6e^{3x}$$
$$f(x) = 6 \times \frac{1}{3}e^{3x} + C$$
$$= 2e^{3x} + C$$

If $f(2) = 2e^6$:
$$2e^6 = 2e^{3\times 2} + C$$
$$= 2e^6 + C$$
$$0 = C$$
So $f(x) = 2e^{3x}$

Anti-derivative of $\frac{1}{x}$

$$\text{If } \frac{dy}{dx} = \frac{1}{x}, \text{ then } y = \ln|x| + C$$

Chain rule

$$\text{If } \frac{dy}{dx} = \frac{f'(x)}{f(x)}, \text{ then } y = \ln|f(x)| + C$$

Proof

$\frac{d}{dx}(\ln x) = \frac{1}{x}$ for $x > 0$, because $\ln x$ is defined only for $x > 0$.

So the anti-derivative of $\frac{1}{x}$ when $x > 0$ is $\ln x$.

Suppose $x < 0$.

Then $\ln(-x)$ is defined because $-x$ is positive.

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$$\frac{d}{dx}[\ln(-x)] = \frac{f'(x)}{f(x)}$$
$$= \frac{-1}{-x}$$
$$= \frac{1}{x}, \qquad x < 0$$

So if $\frac{dy}{dx} = \frac{1}{x}$, then $y = \begin{cases} \ln x + C & \text{if } x > 0 \\ \ln(-x) + C & \text{if } x < 0 \end{cases}$,

or more simply, $\quad y = \ln|x| + C$

EXAMPLE 20

a Find the anti-derivative of $\frac{3}{x}$.

b Find the equation of the function that has $\frac{dy}{dx} = \frac{6x}{x^2 - 5}$ and passes through $(3, 3 \ln 4)$.

Solution

a $\frac{dy}{dx} = \frac{3}{x}$

$= 3 \times \frac{1}{x}$

$y = 3 \ln |x|$

b $\frac{dy}{dx} = \frac{6x}{x^2 - 5}$

$= 3 \times \frac{2x}{x^2 - 5}$

$= 3 \times \frac{f'(x)}{f(x)}$ where $f(x) = x^2 - 5$

$y = 3 \ln f|x| + C$

$= 3 \ln |x^2 - 5| + C$

Substitute $(3, 3 \ln 4)$:

$3 \ln 4 = 3 \ln |3^2 - 5| + C$

$= 3 \ln 4 + C$

$0 = C$

So $y = 3 \ln |x^2 - 5|$

Anti-derivatives of trigonometric functions

If $\frac{dy}{dx} = \cos x$, then $y = \sin x + C$ since $\frac{d}{dx}(\sin x) = \cos x$

If $\frac{dy}{dx} = \sin x$, then $y = -\cos x + C$ since $\frac{d}{dx}(\cos x) = -\sin x$ so $\frac{d}{dx}(-\cos x) = \sin x$

If $\frac{dy}{dx} = \sec^2 x$, then $y = \tan x + C$ since $\frac{d}{dx}(\tan x) = \sec^2 x$

Chain rule

If $\frac{dy}{dx} = \cos(ax + b)$, then $y = \frac{1}{a}\sin(ax + b) + C$

If $\frac{dy}{dx} = \sin(ax + b)$, then $y = -\frac{1}{a}\cos(ax + b) + C$

If $\frac{dy}{dx} = \sec^2(ax + b)$, then $y = \frac{1}{a}\tan(ax + b) + C$

If $\frac{dy}{dx} = f'(x)\cos f(x)$, then $y = \sin f(x) + C$

If $\frac{dy}{dx} = f'(x)\sin f(x)$, then $y = -\cos f(x) + C$

If $\frac{dy}{dx} = f'(x)\sec^2 f(x)$, then $y = \tan f(x) + C$

Proof

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{a}\sin(ax+b)+C\right] &= \frac{1}{a} \times a\cos(ax + b)\\ &= \cos(ax + b)\end{aligned}$$

The other results can be proved similarly.

EXAMPLE 21

a Find the anti-derivative of $\cos 3x$.

b Find the equation of the curve that passes through $\left(\frac{\pi}{4}, 3\right)$ and has $\frac{dy}{dx} = \sec^2 x$.

Solution

a
$$\begin{aligned}y &= \frac{1}{a}\sin(ax + b) + C\\ &= \frac{1}{3}\sin 3x + C\end{aligned}$$

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b $y = \tan x + C$

Substitute $\left(\frac{\pi}{4}, 3\right)$:

$3 = \tan\frac{\pi}{4} + C$

$\quad = 1 + C$

$2 = C$

So $y = \tan x + 2$

Exercise 5.08 Further anti-derivatives

1 Find the anti-derivative of:

a $\sin x$ **b** $\sec^2 x$ **c** $\cos x$

d $\sec^2 7x$ **e** $\sin(2x - \pi)$

2 Anti-differentiate:

a e^x **b** e^{6x} **c** $\frac{1}{x}$

d $\frac{3}{3x-1}$ **e** $\frac{x}{x^2+5}$

3 Find the anti-derivative of:

a $e^x + 5$ **b** $\cos x + 4x$ **c** $x + \frac{1}{x}$

d $8x^3 - 3x^2 + 6x - 3 + x^{-1}$ **e** $\sin 5x - \sec^2 9x$

4 Find the equation of a function with $\frac{dy}{dx} = \cos x$ and passing through $\left(\frac{\pi}{2}, -4\right)$.

5 Find the equation of the function that has $f'(x) = \frac{5}{x}$ and $f(1) = 3$.

6 A function has $\frac{dy}{dx} = 4\cos 2x$ and passes through the point $\left(\frac{\pi}{6}, 2\sqrt{3}\right)$.

Find the exact equation of the function.

7 A curve has $f''(x) = 27e^{3x}$ and has $f(2) = f'(2) = e^6$. Find the equation of the curve.

8 The rate of change of a population over time t years is given by $\frac{dP}{dt} = 1350e^{0.054t}$.
If the initial population is 35 000, find:

a the equation for population

b the population after 10 years

9 The velocity of a particle is given by $\frac{dx}{dt} = 3e^{3t}$ and the particle has an initial displacement of 5 metres. Find the equation for displacement of the particle.

10 A pendulum has acceleration given by $\frac{d^2x}{dt^2} = -9 \sin 3t$, initial displacement 0 cm and initial velocity 3 cm s^{-1}.

a Find the equation for its velocity.

b Find the displacement after 2 seconds.

c Find the times when the pendulum has displacement 0 cm.

EXT1 5.09 Derivative of inverse functions

EXAMPLE 22

Differentiate the inverse function of $y = x^5 + 2$.

Solution

Inverse function:

$$\begin{aligned} x &= y^5 + 2 \\ x - 2 &= y^5 \\ \sqrt[5]{x-2} &= y \\ (x-2)^{\frac{1}{5}} &= y \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{5}(x-2)^{\frac{1}{5}-1} \\ &= \frac{1}{5}(x-2)^{-\frac{4}{5}} \\ &= \frac{1}{5} \times \frac{1}{(x-2)^{\frac{4}{5}}} \\ &= \frac{1}{5\sqrt[5]{(x-2)^4}} \end{aligned}$$

Sometimes it is hard to differentiate inverse functions directly. We can use this property of differentiation:

$\frac{dy}{dx}$ and $\frac{dx}{dy}$

Given that $y = f(x)$ is a differentiable function:

$$\frac{dy}{dx} \times \frac{dx}{dy} = 1 \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

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EXAMPLE 23

Show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ given $y = x^{\frac{1}{3}}$.

Solution

$$y = x^{\frac{1}{3}}$$

$$\frac{dy}{dx} = \frac{1}{3}x^{\frac{1}{3}-1}$$

$$= \frac{1}{3}x^{-\frac{2}{3}}$$

Changing the subject:

$$y = x^{\frac{1}{3}}$$

$$y^3 = x$$

or $x = y^3$

$$\frac{dx}{dy} = 3y^2$$

$$= 3\left(x^{\frac{1}{3}}\right)^2$$

$$= 3x^{\frac{2}{3}}$$

$$\frac{dy}{dx} \times \frac{dx}{dy} = \frac{1}{3}x^{-\frac{2}{3}} \times 3x^{\frac{2}{3}}$$

$$= 1$$

We can use this property to find the derivative of inverse functions.

EXAMPLE 24

a Differentiate the inverse function of $y = x^3 - 1$, leaving your answer in terms of y.

b Find the gradient of the tangent at the point (7, 2) on the inverse function.

Solution

a Inverse function:

$$x = y^3 - 1$$

$$\frac{dx}{dy} = 3y^2$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{3y^2}$$

b At (7, 2):

$$\frac{dy}{dx} = \frac{1}{3y^2}$$

$$= \frac{1}{3 \times 2^2}$$

$$= \frac{1}{12}$$

So the gradient of the tangent at (7, 2) on the inverse function is $\frac{1}{12}$.

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EXAMPLE 25

a Find the derivative of the inverse function $f^{-1}(x)$ of $f(x) = x(x + 1)^4$ in terms of y.

b Given that $f^{-1}(-1) = 2$, find the gradient of the tangent to $y = f^{-1}(x)$ at this point.

Solution

a Inverse function is $x = y(y + 1)^4$.

It is difficult to change the subject of this equation to y, so we find $\frac{dx}{dy}$.

$$\frac{dx}{dy} = u'v + v'u \text{ where } u = y \text{ and } v = (y + 1)^4$$

$$u' = 1 \qquad v' = 4(y + 1)^3$$

$$\begin{aligned}\frac{dx}{dy} &= 1 \times (y + 1)^4 + 4(y + 1)^3 \times y \\ &= (y + 1)^4 + 4y(y + 1)^3 \\ &= (y + 1)^3 (y + 1 + 4y) \\ &= (y + 1)^3(5y + 1)\end{aligned}$$

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\frac{dx}{dy}} \\ &= \frac{1}{(y+1)^3(5y+1)}\end{aligned}$$

b Since $f^{-1}(-1) = 2$, the curve passes through $(-1, 2)$.

Substitute $y = 2$:

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{(2+1)^3(5\times 2+1)} \\ &= \frac{1}{3^3 \times 11} \\ &= \frac{1}{297}\end{aligned}$$

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EXT1 Exercise 5.09 Derivative of inverse functions

1 Show that $\frac{dy}{dx} \times \frac{dx}{dy} = 1$ given:

a $y = 4x + 3$ **b** $y = x^3$ **c** $y = e^x$

d $y = \ln x$ **e** $y = x^7 - 1$

2 Differentiate $f^{-1}(x)$ given:

a $f(x) = e^x$ **b** $f(x) = \ln x$ **c** $f(x) = \sqrt{x}$

d $f(x) = x^7 - 1$ **e** $f(x) = (x + 2)^3$

3 Find the gradient of the tangent to the inverse function at:

a $(5, 1)$ given $f(x) = x^3 + 4$ **b** $(-1, 1)$ given $f(x) = 2x - 3$

c $(1, 0)$ given $f(x) = e^{3x}$ **d** $(2, 5)$ given $f(x) = \sqrt{x-1}$

e $\left(\frac{1}{9}, 2\right)$ given $f(x) = \frac{1}{x^3+1}$

4 a Find the derivative of the inverse function f^{-1} given $f(x) = 4x^3$.

b The point $(4, 1)$ lies on f^{-1}. Find the gradient of:

i the tangent **ii** the normal at that point

5 a By restricting $f(x)$ to a monotonic increasing domain, find the inverse function of $f(x) = x^2 + 1$.

b Find the derivative of the inverse function f^{-1}.

c Given that $(5, 2)$ lies on f^{-1}, find the gradient of the tangent at this point.

6 Find $\frac{dx}{dy}$ of the inverse function $f^{-1}(x)$ of each function in terms of y.

a $f(x) = x^2 e^x$ **b** $f(x) = 3x \sin 2x$ **c** $y = x(2x - 3)^4$

d $f(x) = \frac{3x-1}{2x+5}$ **e** $y = \frac{\ln x}{x+2}$

7 Find the gradient of the tangent at each point given on the inverse function of:

a $f(x) = (3x + 1)(x - 4)^5$ at $(-10, 3)$

b $y = (x - 3)\cos x$ at $(-2, 0)$

c $f(x) = \frac{x^3}{3x-4}$ at $(4, 2)$

d $y = x \ln x$ at $(0, 1)$

e $y = \frac{\sin 3x}{x^2}$ at $\left(-\frac{4}{\pi^2}, \frac{\pi}{2}\right)$

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Derivatives of inverse trigonometric functions

Inverse trigonometric functions and gradient

EXT1 5.10 Derivative of inverse trigonometric functions

Derivative of $\sin^{-1} x$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

Proof

Let $y = \sin^{-1} x$

Then $x = \sin y$

$$\frac{dx}{dy} = \cos y$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$= \frac{1}{\cos y}$$

Using $\sin^2 \theta + \cos^2 \theta = 1$:

$$\cos^2 y = 1 - \sin^2 y$$

$$\cos y = \sqrt{1-\sin^2 y}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$

$$= \frac{1}{\sqrt{1-\sin^2 y}}$$

$$= \frac{1}{\sqrt{1-x^2}}$$

EXAMPLE 26

Find the equation of the tangent to the curve $y = \sin^{-1} x$ at the point $\left(0, \frac{\pi}{2}\right)$.

Solution

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

At $\left(0, \frac{\pi}{2}\right)$:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-0^2}}$$

$$= 1$$

So $m = 1$

Equation:

$$y - y_1 = m(x - x_1)$$

$$y - \frac{\pi}{2} = 1(x - 0)$$

$$= x$$

$$2y - \pi = 2x$$

$$0 = 2x - 2y + \pi$$

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You can use the chain rule to differentiate.

EXAMPLE 27

Differentiate $\sin^{-1}(5x-1)$.

Solution

$y=\sin^{-1}(5x-1)$ is a composite function.

Let $y=\sin^{-1}u$ where $u=5x-1$.

$$\frac{dy}{du}=\frac{1}{\sqrt{1-u^2}} \quad \text{and} \quad \frac{du}{dx}=5$$

$$\begin{aligned}\frac{dy}{dx}&=\frac{dy}{du}\times\frac{du}{dx}\\&=\frac{1}{\sqrt{1-u^2}}\times 5\\&=\frac{5}{\sqrt{1-u^2}}\\&=\frac{5}{\sqrt{1-(5x-1)^2}}\\&=\frac{5}{\sqrt{1-(25x^2-10x+1)}}\\&=\frac{5}{\sqrt{-25x^2+10x}}\end{aligned}$$

There is a simplified chain rule for differentiating $\sin^{-1}\left(\frac{x}{a}\right)$.

Chain rule

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right]=\frac{1}{\sqrt{a^2-x^2}}$$

The proof is similar to the proof of the derivative of $\sin^{-1}x$.

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EXAMPLE 28

Differentiate $\sin^{-1}\left(\frac{x}{3}\right)$.

Solution

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{\sqrt{a^2 - x^2}} \text{ where } a = 3\\ &= \frac{1}{\sqrt{3^2 - x^2}}\\ &= \frac{1}{\sqrt{9 - x^2}}\end{aligned}$$

Derivative of $\cos^{-1} x$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1 - x^2}}$$

Chain rule

$$\frac{d}{dx}\left[\cos^{-1}\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2 - x^2}}$$

The proofs of these results are similar to the proof of $\sin^{-1} x$.

Derivative of inverse cosine function

EXAMPLE 29

Differentiate:

a $\cos^{-1}\left(\frac{x}{7}\right)$

b $\cos^{-1} 2x$

Solution

a $$\begin{aligned}\frac{dy}{dx} &= -\frac{1}{\sqrt{a^2 - x^2}} \text{ where } a = 7\\ &= -\frac{1}{\sqrt{7^2 - x^2}}\\ &= -\frac{1}{\sqrt{49 - x^2}}\end{aligned}$$

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b $\cos^{-1} 2x = \cos^{-1}\left(\dfrac{x}{\frac{1}{2}}\right)$

Method 1: Chain rule

$\cos^{-1} 2x$ is a composite function.

Let $y = \cos^{-1} u$ where $u = 2x$

$$\frac{dy}{du} = -\frac{1}{\sqrt{1-u^2}} \quad \text{and} \quad \frac{du}{dx} = 2$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\
&= -\frac{1}{\sqrt{1-u^2}} \times 2 \\
&= -\frac{2}{\sqrt{1-(2x)^2}} \\
&= -\frac{2}{\sqrt{1-4x^2}}
\end{aligned}$$

Method 2: Formula

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{1}{\sqrt{a^2-x^2}} \text{ where } a = \frac{1}{2} \\
&= -\frac{1}{\sqrt{\left(\frac{1}{2}\right)^2 - x^2}} \\
&= -\frac{1}{\sqrt{\frac{1}{4} - x^2}} \\
&= -\frac{1}{\sqrt{\frac{1-4x^2}{4}}} \\
&= -\frac{1}{\frac{\sqrt{1-4x^2}}{2}} \\
&= -\frac{2}{\sqrt{1-4x^2}}
\end{aligned}$$

Derivative of $\tan^{-1} x$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

Proof

Let $y = \tan^{-1} x$

Then $x = \tan y$

$$\frac{dx}{dy} = \sec^2 y$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$

Using $\tan^2 \theta + 1 = \sec^2 \theta$:

$$\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$$

$$= \frac{1}{1+x^2}$$

Chain rule

$$\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2+x^2}$$

The proof is similar to the proof of the derivative of $\tan^{-1} x$.

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EXAMPLE 30

a Find the gradient of the normal to the curve $y = \tan^{-1} x$ at the point where $x = \frac{1}{\sqrt{3}}$.

b Differentiate $\tan^{-1}\left(\frac{x}{5}\right)$.

Solution

a $\frac{dy}{dx} = \frac{1}{1+x^2}$

When $x = \frac{1}{\sqrt{3}}$

$$\frac{dy}{dx} = \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2}$$

$$= \frac{1}{1+\frac{1}{3}}$$

$$= \frac{1}{\frac{3}{3}+\frac{1}{3}}$$

$$= \frac{1}{\frac{4}{3}}$$

$$= \frac{3}{4}$$

So $m_1 = \frac{3}{4}$

Gradient is perpendicular to the normal.

$m_1 m_2 = -1$

$\frac{3}{4} m_2 = -1$

$m_2 = -1 \times \frac{4}{3}$

$= -\frac{4}{3}$

b $\frac{dy}{dx} = \frac{a}{a^2+x^2}$ where $a = 5$

$$= \frac{5}{5^2+x^2}$$

$$= \frac{5}{25+x^2}$$

EXT1 Exercise 5.10 Derivative of inverse trigonometric functions

1 Differentiate:

a $\cos^{-1} x$ **b** $2 \sin^{-1} x$ **c** $\tan^{-1} x$

d $\cos^{-1} 3x$ **e** $4 \sin^{-1} 2x$ **f** $\sin^{-1} (x^2)$

g $\tan^{-1} (2x - 1)$ **h** $5 \cos^{-1} 8x$ **i** $\cos^{-1} \left(\frac{x}{3}\right)$

j $\tan^{-1} \left(\frac{x}{2}\right)$

2 For each function, find the gradient of:

i the tangent **ii** the normal

a $y = \cos^{-1} x$ at the point $\left(0, \frac{\pi}{2}\right)$

b $y = \tan^{-1} (2x)$ at the point where $x = \frac{1}{4}$

c $f(x) = (\sin^{-1} x)^3$ at the point where $x = \frac{1}{2}$

d $y = \cos^{-1} \left(\frac{x}{3}\right)$ at the point $\left(0, \frac{\pi}{2}\right)$

e $y = \tan^{-1} \left(\frac{x}{5}\right)$ at the point where $x = 0$

3 Find the equation of the tangent to the curve $y = \sin^{-1} (2x)$ at the point where $x = 0$.

4 Find the equation of the normal to the curve $y = \tan^{-1} 5x$ at $\left(\frac{1}{5}, \frac{\pi}{4}\right)$.

5 Find the derivative of:

a $3 \sin^{-1} \left(\frac{x}{6}\right)$ **b** $3 \cos^{-1} \sqrt{x}$ **c** $\cos^{-1} \left(\frac{x}{7}\right)$

d $5 \sin^{-1} (3x + 2)$ **e** $x \cos^{-1} x$ **f** $(\tan^{-1} x + 1)^5$

6 Differentiate:

a $\sin^{-1} (\cos x)$ **b** $\cos^{-1} (\cos x)$ **c** $\sin^{-1} (\ln x)$

d $\tan^{-1} (e^x)$ **e** $\ln (\sin^{-1} x)$ **f** $\frac{1}{\tan^{-1} x}$

g $\tan^{-1} (\cos^{-1} x + 1)$ **h** $\tan^{-1} \left(\frac{1}{x}\right)$ **i** $\sin^{-1} \left(\frac{x}{2} + 1\right)$

j $e^{\cos^{-1} x}$

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7 Show that $\frac{d}{dx}(\sin^{-1} x + \cos^{-1} x) = 0$.

8 Find the second derivative of:

a $\cos^{-1}\left(\frac{x}{3}\right)$ **b** $\ln(\tan^{-1} x)$

9 Find the equation of the tangent to the curve $y = \sin^{-1} x$ at the point where $x = -\frac{1}{2}$.

10 a Find $\frac{d}{dx}\left[\tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)\right]$.

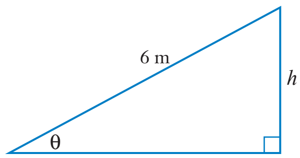
b Draw the graph of $y = \tan^{-1} x + \tan^{-1}\left(\frac{1}{x}\right)$.

11 Differentiate:

a $\cos^{-1}(e^{2x})$ **b** $\ln(\tan^{-1} x)$ **c** $\tan^{-1}(\ln x)$

d $\sin^{-1}\sqrt{1-x^2}$ **e** $e^{\tan^{-1} x}$

12 A 6 metre long ladder is leaning against a wall at a height of h and angle θ as shown.



a Show that $\theta = \sin^{-1}\left(\frac{h}{6}\right)$.

b The ladder slips down the wall at a constant rate of 0.05 m s^{-1}. Find the rate at which the angle is changing when the height is 2.5 m.

13 A hot air balloon rises into the air at 2 metres per second. Jan is standing 100 m away from the balloon.

a What is the height of the balloon after t seconds?

b If the angle of elevation from Jan up to the balloon is θ, write an equation for θ in terms of t.

c Find the rate of change in θ (in radians) after:

i 5 seconds **ii** one minute

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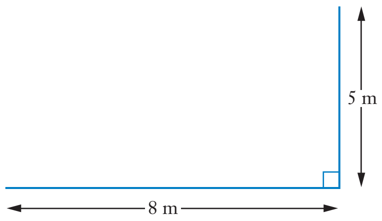
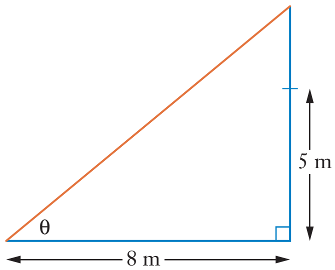
14 Two walls along a property are 8 m and 5 m long as shown.



A builder extends the 5 m wall as shown at 0.5 metres per minute.



a Write an equation for the angle θ in terms of t.

b Find the rate at which θ is changing after:

i 5 minutes **ii** 20 minutes **iii** an hour

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Summary of differentiation rules

Rule	Chain rule
$\frac{d}{dx}(x^n) = nx^{n-1}$	$\frac{d}{dx}[f(x)]^n = f'(x)n[f(x)]^{n-1}$
$\frac{d}{dx}(e^x) = e^x$	$\frac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)}$
$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$
$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}[\cos f(x)] = -f'(x)\sin f(x)$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}[\tan f(x)] = f'(x)\sec^2 f(x)$
EXT1 $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	EXT1 $\frac{d}{dx}\left[\sin^{-1}\left(\frac{x}{a}\right)\right] = \frac{1}{\sqrt{a^2-x^2}}$
EXT1 $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$	EXT1 $\frac{d}{dx}\left[\cos^{-1}\left(\frac{x}{a}\right)\right] = -\frac{1}{\sqrt{a^2-x^2}}$
EXT1 $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	EXT1 $\frac{d}{dx}\left[\tan^{-1}\left(\frac{x}{a}\right)\right] = \frac{a}{a^2+x^2}$

Product rule: $\frac{d}{dx}(uv) = u'v + v'u$

Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - v'u}{v^2}$

5. TEST YOURSELF

Practice quiz

For Questions **1** to **4**, choose the correct answer **A**, **B**, **C** or **D**.

1 The anti-derivative of sin $6x$ is:

A $\frac{1}{6}\cos 6x$ **B** $6\cos 6x$ **C** $-6\cos 6x$ **D** $-\frac{1}{6}\cos 6x$

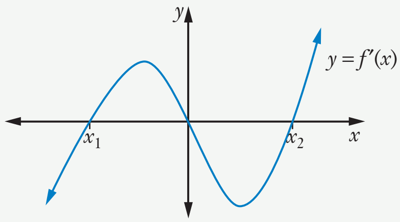
2 The gradient of the tangent to the curve $y = e^{2x} + x$ at $(0, 1)$ is:

A 2 **B** 3 **C** e **D** 1

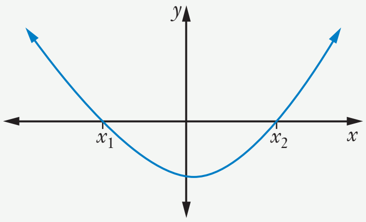
3 If $y = \cos 2x$, then $\frac{d^2y}{dx^2}$ is :

A $-4y$ **B** $-2y$ **C** $4y$ **D** y

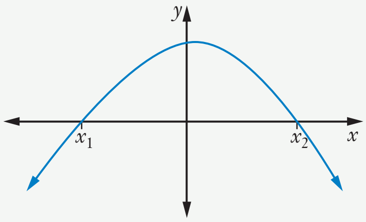
4 The graph of the derivative $y = f'(x)$ is shown. Which of the following graphs could be the graph of $y = f(x)$?



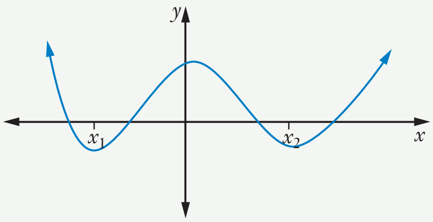
A



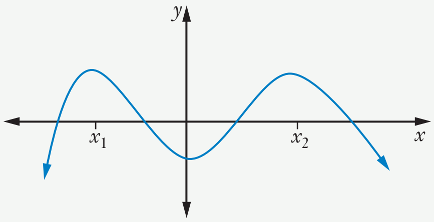
B



C



D



5 Differentiate:

a e^{5x} **b** $2e^{1-x}$ **c** $\ln 4x$ **d** $\ln(4x + 5)$

e xe^x **f** $\frac{\ln x}{x}$ **g** $(e^x + 1)^{10}$

6 Differentiate:

a $\cos x$ **b** $2\sin x$ **c** $\tan x + 1$ **d** $x\sin x$

e $\frac{\tan x}{x}$ **f** $\cos 3x$ **g** $\tan 5x$

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7 Find the equation of the tangent to the curve $y = 2 + e^{3x}$ at the point where $x = 0$.

8 Find the equation of the tangent to the curve $y = \sin 3x$ at the point $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.

9 If $x = \cos 2t$, show that $\frac{d^2x}{dt^2} = -4x$.

10 EXT1 Differentiate:

a $\sin^{-1} x$ **b** $\tan^{-1} 3x$ **c** $2\cos^{-1} 5x$

11 Find the exact gradient of the normal to the curve $y = x - e^{-x}$ at the point where $x = 2$.

12 Find the anti-derivative of:

a $10x^4 - 4x^3 + 6x - 3$ **b** e^{5x} **c** $\sec^2 9x$

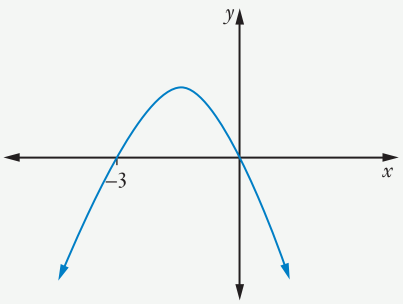
d $\frac{1}{x+5}$ **e** $\cos 2x$ **f** $\sin\left(\frac{x}{4}\right)$

13 Find the gradient of the tangent to the curve $y = 3\cos 2x$ at the point where $x = \frac{\pi}{6}$.

14 A curve has $\frac{dy}{dx} = 6x^2 + 12x - 5$. If the curve passes through the point $(2, -3)$, find the equation of the curve.

15 EXT1 Find the derivative of $f^{-1}(x)$ if $f(x) = x^5 + 3$.

16 Sketch the graph of the anti-derivative of the following function, given that the anti-derivative passes through (0, 4).



17 Find the equation of the normal to the curve $y = \ln x$ at the point $(2, \ln 2)$.

18 Find the equation of the normal to the curve $y = \tan x$ at the point $\left(\frac{\pi}{4}, 1\right)$.

19 EXT1 Find the derivative of:

a $\sin^{-1} x$ **b** $\cos^{-1}\left(\frac{x}{5}\right)$ **c** $\tan^{-1} x$

d $\sin^{-1} 4x$ **e** $\tan^{-1}\left(\frac{x}{2}\right)$

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20 Differentiate:

a $(5x^2 + 7)^4$ **b** $4x(2x - 3)^7$ **c** $\dfrac{5x - 1}{3x + 4}$

d $2x^3e^x$ **e** $\dfrac{\tan 3x}{x + 1}$

EXT1 **21** **a** Evaluate $\dfrac{d}{dx}(\sin^{-1} x + \cos^{-1} x)$.

b Explain this result.

22 If $f''(x) = 15x + 12$ and $f(2) = f'(2) = 5$, find the equation of $y = f(x)$.

23 If $f(x) = 3x^5 - 2x^4 + x^3 - 2$, find:

a $f(-1)$ **b** $f'(-1)$ **c** $f''(-1)$

EXT1 **24** **a** Find the inverse function $f^{-1}(x)$ given $f(x) = \sqrt{x + 1}$.

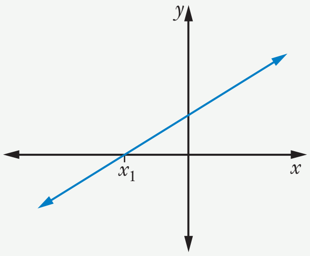
b Find the point P on $f^{-1}(x)$ where $x = 3$.

c Find the equation of the tangent to $f^{-1}(x)$ at P.

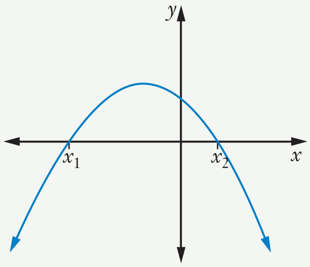
25 EXT1 Differentiate $x \tan^{-1} x$.

26 Sketch an example of the graph of an anti-derivative function for each graph.

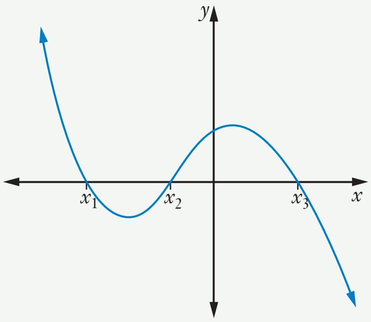
a



b



c



27 A function has $f'(3) = 5$ and $f(3) = 2$. If $f''(x) = 12x - 6$, find the equation of the function.

28 EXT1 Find the equation of the tangent to the curve $y = \sin^{-1}\left(\dfrac{x}{3}\right)$ at the point $\left(1\dfrac{1}{2}, \dfrac{\pi}{6}\right)$.

29 Find the anti-derivative of:

a $x^3(3x^4 - 5)^6$ **b** $3x(x^2 + 1)^9$

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5. CHALLENGE EXERCISE

1 Find the exact gradient of the tangent to the curve $y = e^{x + \ln x}$ at the point where $x = 1$.

EXT1 **2** **a** Show that $\tan^{-1}\left(\frac{4}{5}\right) + \tan^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$.

b Find $\frac{d}{dx}\left(\tan^{-1} x + \tan^{-1}\frac{1}{x}\right)$.

c Show that $\tan^{-1} x + \tan^{-1}\frac{1}{x} = \frac{\pi}{2}$ for all x.

3 Find the first and second derivatives of $\frac{5 - x}{(4x^2 + 1)^3}$.

4 Find the anti-derivative of:

a $2xe^{x^2}$ **b** $x^2 \sin(x^3)$

5 Differentiate $e^{x \sin 2x}$.

6 A curve passes through the point $(0, -1)$ and the gradient at any point is given by $(x + 3)(x - 5)$. Find the equation of the curve.

7 EXT1 Differentiate:

a $\sin^{-1}(x^2)$ **b** $\tan^{-1}(e^x)$ **c** $\ln(\sin x + \cos x)$

8 The rate of change of V with respect to t is given by $\frac{dV}{dt} = (2t - 1)^2$.
If $V = 5$ when $t = \frac{1}{2}$, find V when $t = 3$.

9 Find the derivative of $y = \frac{x \log_e x}{e^x}$.

10 EXT1 A car is stopped at point A, 20 km south of an intersection O. Another car leaves the intersection and travels east at 80 km h^{-1}. If this car is at point B:

a Find an equation for angle OAB after t hours.

b Find the rate at which angle OAB is changing after 2 hours (in degrees and minutes per hour, to the nearest minute).

EXT1 **11** **a** Find the inverse function f^{-1} in terms of y given $f(x) = x + e^x$.

b Find the image P on f^{-1} of the point where $f(1) = 1 + e$.

c Find the equation of the tangent to f^{-1} at P.

12 **a** Differentiate $\ln(\tan x)$.

b Find the anti-derivative of $\tan x$.

13 Find the anti-derivative of:

a $x^2 \sin(x^3 - \pi)$ **b** xe^{x^2}

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