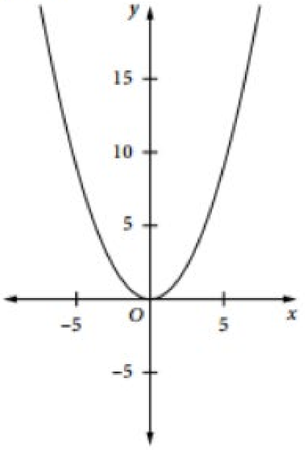
CHAPTER 15
Graphing techniques

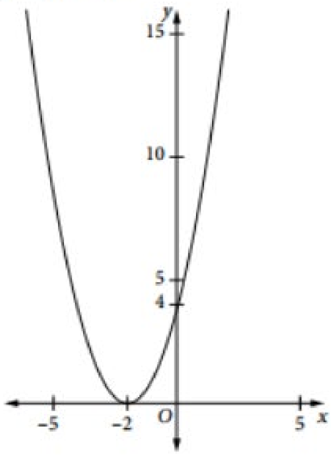
15.1 TRANSFORMATION OF GRAPHS USING $y = f(x + b)$ AND $y = f(x) + c$

Consider the following graphs:

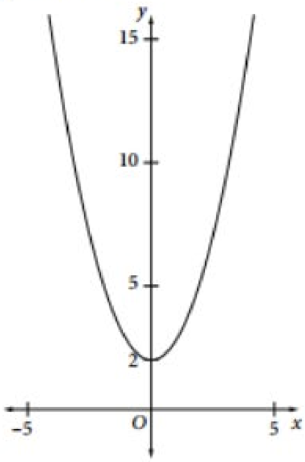
$y = x^2$



$y = (x + 2)^2$



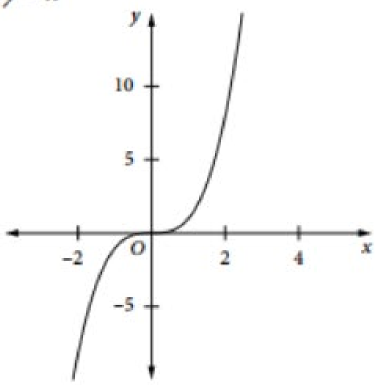
$y = x^2 + 2$



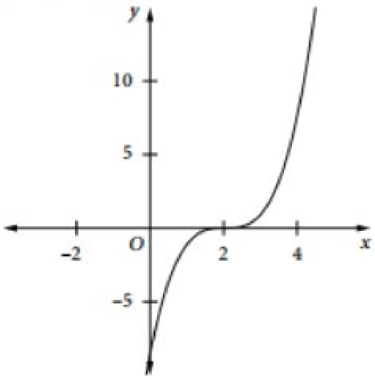
If $y = x^2$ is written as $y = f(x)$ then $y = (x + 2)^2$ becomes $y = f(x + 2)$ and $y = x^2 + 2$ becomes $y = f(x) + 2$.
In $y = f(x + 2)$ the curve for $y = f(x)$ has been moved 2 units to the left, $(0, 0)$ moved to $(-2, 0)$.
In $y = f(x) + 2$ the curve for $y = f(x)$ has been moved 2 units upwards, $(0, 0)$ moved to $(0, 2)$.

Now consider the following similar graphs:

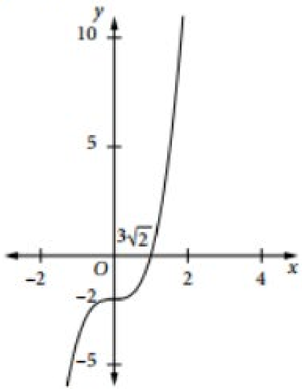
$y = x^3$



$y = (x - 2)^3$



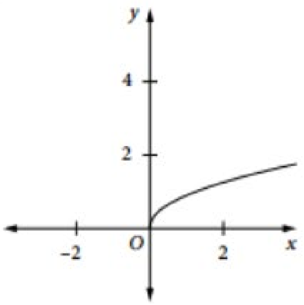
$y = x^3 - 2$



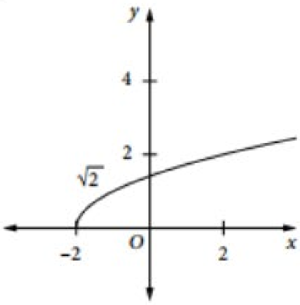
If $y = x^3$ is written as $y = f(x)$ then $y = (x - 2)^3$ becomes $y = f(x - 2)$ and $y = x^3 - 2$ becomes $y = f(x) - 2$.
In $y = f(x - 2)$ the curve for $y = f(x)$ has been moved 2 units to the right, $(0, 0)$ moved to $(2, 0)$.
In $y = f(x) - 2$ the curve for $y = f(x)$ has been moved 2 units downwards, $(0, 0)$ moved to $(0, -2)$.

In the cases above, the same function is translated horizontally and vertically by changing the function. The same changes may also be applied to functions involving square roots, for example:

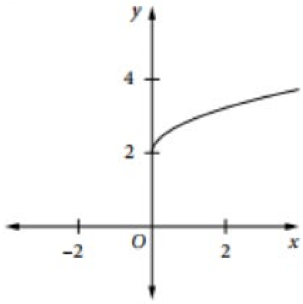
$y = \sqrt{x}$



$y = \sqrt{x+2}$



$y = \sqrt{x} + 2$



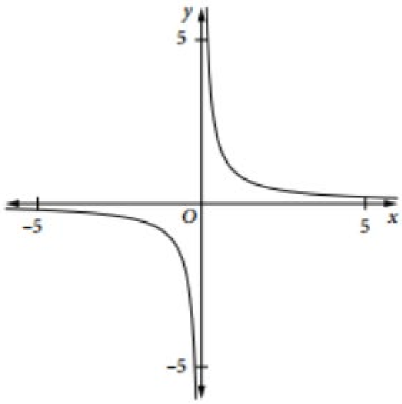
If $y = \sqrt{x}$ is written as $y = f(x)$ then $y = \sqrt{x+2}$ becomes $y = f(x + 2)$ and $y = \sqrt{x} + 2$ becomes $y = f(x) + 2$.

In $y = f(x + 2)$ the curve for $y = f(x)$ has been moved 2 units to the left, (0, 0) moved to (−2, 0).

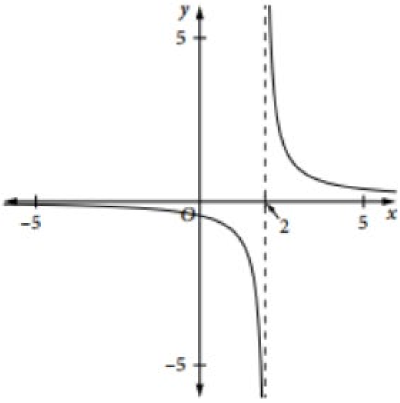
In $y = f(x) + 2$ the curve for $y = f(x)$ has been moved 2 units upwards, (0, 0) moved to (0, 2).

Similarly, consider the following reciprocal functions:

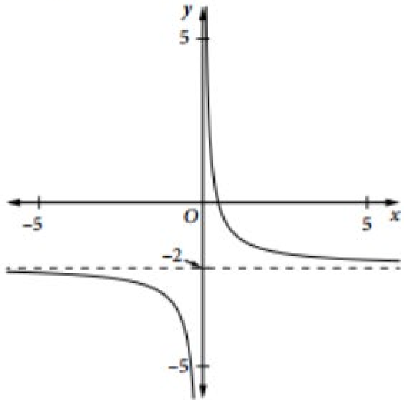
$y = \frac{1}{x}$



$y = \frac{1}{x-2}$



$y = \frac{1}{x} - 2$



If $y = \frac{1}{x}$ is written as $y = f(x)$ then $y = \frac{1}{x-2}$ becomes $y = f(x - 2)$ and $y = \frac{1}{x} - 2$ becomes $y = f(x) - 2$.

In $y = f(x - 2)$ the curve for $y = f(x)$ has been moved 2 units to the right.

In $y = f(x) - 2$ the curve for $y = f(x)$ has been moved 2 units downwards.

Hence the graph of $y = f(x + b)$ is just the graph of $y = f(x)$ moved b units to the left.

Also the graph of $y = f(x) + c$ is just the graph of $y = f(x)$ moved c units upwards.

MAKING CONNECTIONS

Horizontal translation of the graph of a function

Move the slider to explore the effect of changing b on the graph of a function $f(x + b)$.

MAKING CONNECTIONS

Vertical translation of the graph of a function

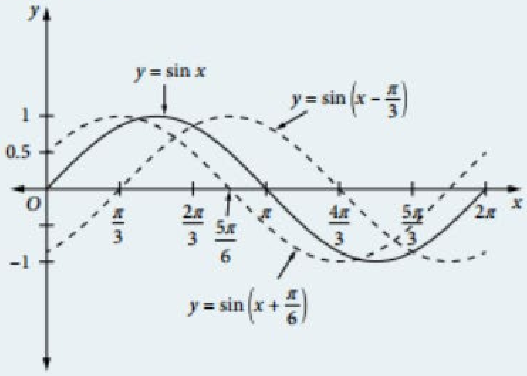
Move the slider to explore the effect of changing c on the graph of a function $f(x) + c$.

Example 1

On the same set of axes, for $0 \leq x \leq 2\pi$, draw the graphs of:

(a) $y = \sin x$ **(b)** $y = \sin\left(x + \frac{\pi}{6}\right)$ **(c)** $y = \sin\left(x - \frac{\pi}{3}\right)$

Solution



EXERCISE 15.1 TRANSFORMATION OF GRAPHS USING $y = f(x + b)$ AND $y = f(x) + c$

For the following questions, graphs may be drawn using a table of values or appropriate graphing software.

1 On the same diagram, draw the graphs of:

(a) $y = x^2, y = (x-2)^2, y = x^2 - 2$

(b) $y = \sqrt{x}, y = \sqrt{x-4}, y = \sqrt{x} - 4$

(c) $y = \frac{1}{x}, y = \frac{1}{x+3}, y = \frac{1}{x} + 3$

(d) $y = \cos x, y = \cos\left(x + \frac{\pi}{6}\right), y = \cos\left(x - \frac{\pi}{3}\right)$ for $0 \leq x \leq 2\pi$.

2 On the same diagram, draw the graphs of:

(a) $y = e^x, y = e^{x+2}, y = e^x + 2$

(b) $y = \ln x, y = \ln(x - e), y = \ln x - e$

(c) $y = e^{-x}, y = e^{-x-1}, y = e^{-x} - 1$

(d) $y = \ln(-x), y = \ln(1 - x), y = \ln(-x) + 1$

3 On the same diagram, draw the following graphs for $0 \leq x \leq 2\pi$:

(a) $y = \cos x, y = \cos\left(x - \frac{\pi}{6}\right), y = \cos\left(x + \frac{\pi}{3}\right)$

(b) $y = \tan x, y = \tan\left(x + \frac{\pi}{4}\right), y = \tan\left(x - \frac{\pi}{4}\right)$

(c) $y = \operatorname{cosec} x, y = \operatorname{cosec}\left(x + \frac{\pi}{4}\right), y = \operatorname{cosec}\left(x - \frac{\pi}{2}\right)$

(d) $y = \cos x, y = \cos x + 2, y = \cos x - 1$

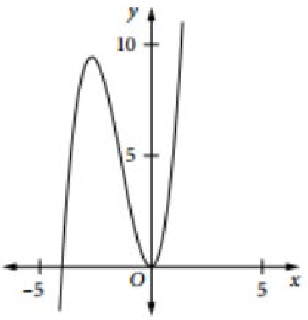
4 On the same diagram, draw the graphs of:

(a) $y = \frac{1}{x^2}, y = \frac{1}{(x-1)^2}, y = \frac{1}{x^2} - 1$

(b) $y = x^3 + x, y = (x-1)^3 + (x-1), y = x^3 + x - 1$

(c) $y = \sqrt{x^2}, y = \sqrt{(x+2)^2}, y = \sqrt{x^2} + 2$

5 The diagram below shows the graph of $y = f(x)$. Which of the options shows the graph of $y = f(x - 1)$?



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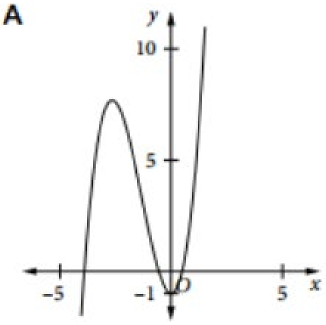

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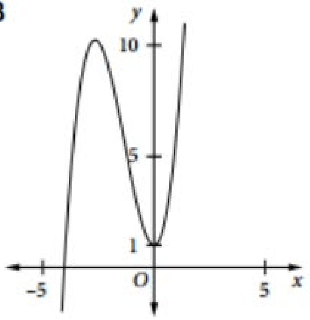

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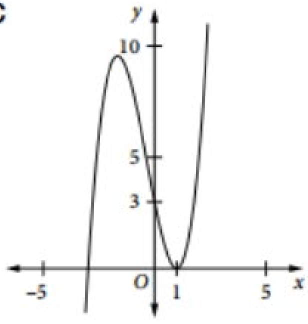

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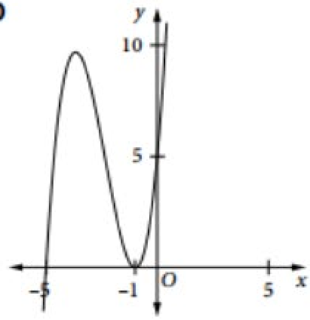
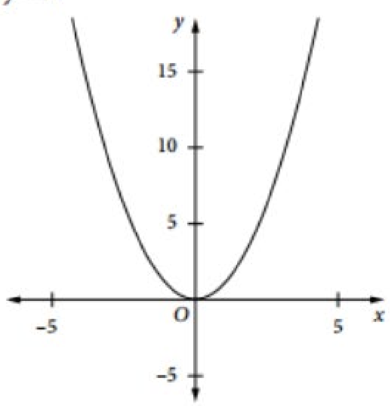


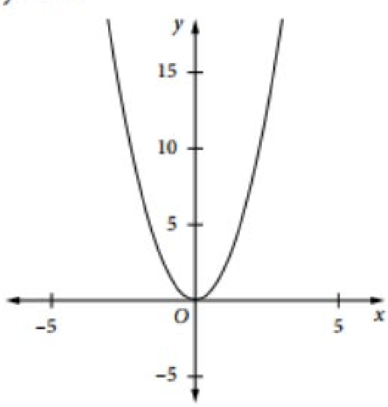
15.2 TRANSFORMATION OF GRAPHS USING $y = kf(x)$ AND $y = kf(x + b)$

Consider the following graphs:

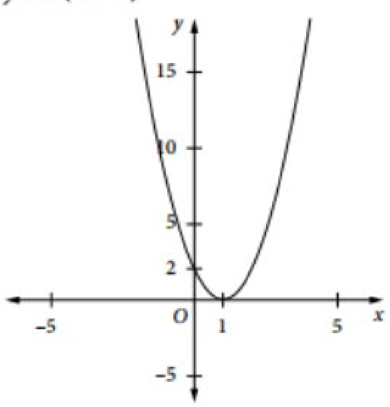
$y = x^2$



$y = 2x^2$



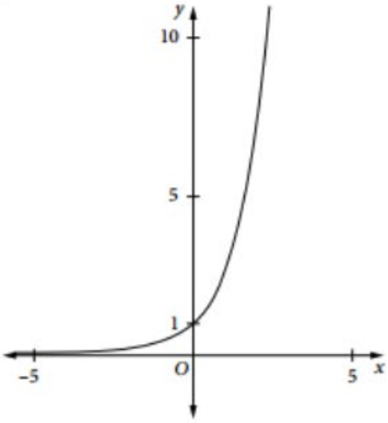
$y = 2(x - 1)^2$



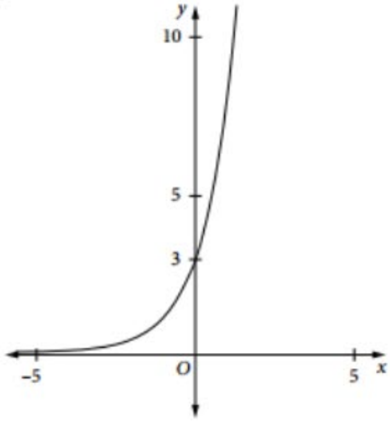
If $y = x^2$ is written as $y = f(x)$ then $y = 2x^2$ becomes $y = 2f(x)$ and $y = 2(x - 1)^2$ becomes $y = 2f(x - 1)$.
In $y = 2f(x)$ the curve for $y = f(x)$ has stretched (dilated) by a factor of 2 from the x-axis.
In $y = 2f(x - 1)$ the curve for $y = f(x)$ has been moved 1 unit to the right and then stretched by a factor of 2 from the x-axis.

Now consider the following exponential graphs:

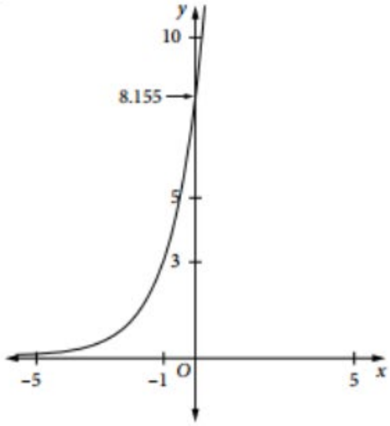
$y = e^x$



$y = 3e^x$



$y = 3e^{x+1}$



If $y = e^x$ is written as $y = f(x)$ then $y = 3e^x$ becomes $y = 3f(x)$ and $y = 3e^{x+1}$ becomes $y = 3f(x + 1)$.

In $y = 3f(x)$ the curve for $y = f(x)$ has stretched (dilated) by a factor of 3 from the x-axis.

In $y = 3f(x + 1)$ the curve for $y = f(x)$ has been moved 1 unit to the left and then been stretched by a factor of 3 from the x-axis.

In all these cases, the graph of $y = kf(x)$ is just the graph of $y = f(x)$ stretched (dilated) by a factor of k.

As for the cases in the previous section, the graph of $y = kf(x + b)$ is just the graph of $y = f(x)$ stretched (dilated) by a factor of k and also translated horizontally b units to the left.

MAKING CONNECTIONS

Dilation from the *x*-axis of the graph of a function

Move the sliders to explore the effects of changing k and b on the graph of a function $kf(x + b)$.

EXERCISE 15.2 TRANSFORMATION OF GRAPHS USING $y = kf(x)$ AND $y = kf(x + b)$

For the following questions, graphs may be drawn using a table of values or appropriate graphing software.

1 On the same diagram, draw the graph of each equation, stating the dilation factor.

(a) $y = x^2, y = 3x^2, y = 3(x + 2)^2$

(b) $y = x^3, y = 2x^3, y = 2(x - 1)^3$

(c) $y = \sqrt{x}, y = 5\sqrt{x}, y = 5\sqrt{x - 4}$

(d) $y = x^2, y = \frac{x^2}{4}, y = \frac{(x - 2)^2}{4}$

2 On the same diagram, draw the graph of each equation, stating the dilation factor.

(a) $y = e^x, y = \frac{e^x}{2}, y = \frac{e^{x-1}}{2}$

(b) $y = \sin x, y = 2\sin x, y = 2\sin\left(x - \frac{\pi}{2}\right)$ for $-\pi \le x \le \pi$

(c) $y = \sec x, y = \frac{\sec x}{2}, y = \frac{\sec\left(x - \frac{\pi}{6}\right)}{2}$ for $-\pi \le x \le \pi$

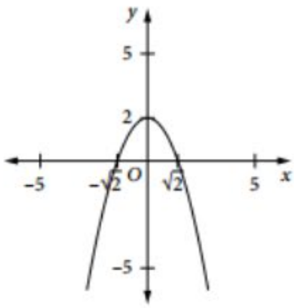
3 On the same diagram, draw the graph of each equation, stating the dilation factor.

(a) $y = x^2 + x, y = 2(x^2 + x), y = 2[(x - 1)^2 + (x - 1)]$

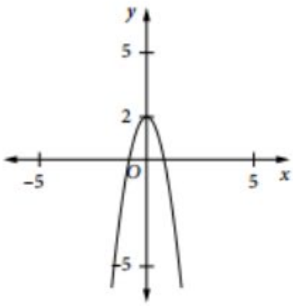
(b) $y = -x^2, y = -3x^2, y = -3(x + 2)^2$

(c) $y = x - x^3, y = 2(x - x^3), y = 2[(x + 1) - (x + 1)^3]$

4 The diagram to the right shows the graph of $y = f(x)$.
Which diagram below shows the graph of $y = 3f(x)$?



A



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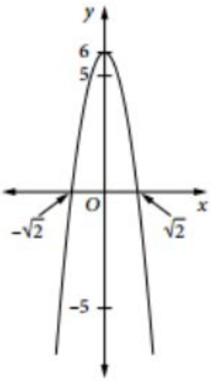

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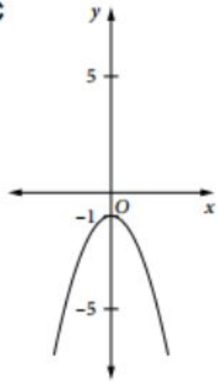

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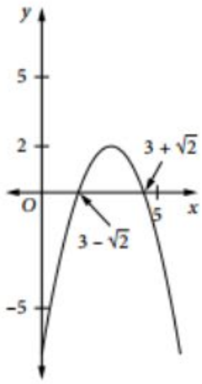
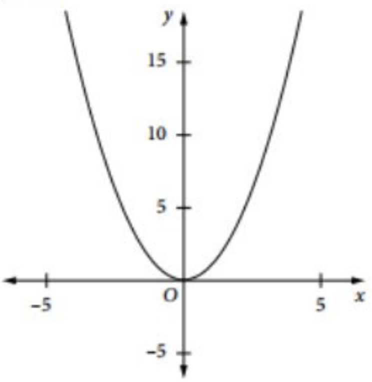


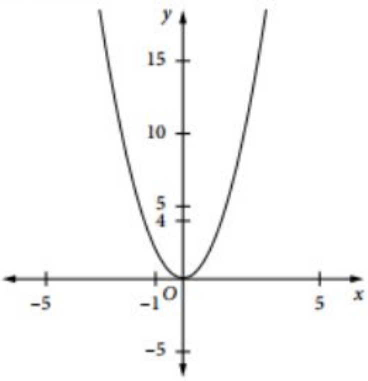
15.3 TRANSFORMATION OF GRAPHS USING $y = f(ax)$ AND $y = f(a(x + b))$

Consider the following graphs.

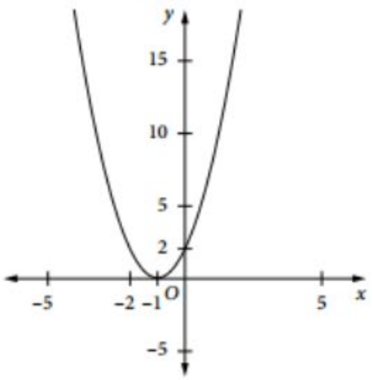
$y = x^2$



$y = (2x)^2$



$y = (2(x + 1))^2$



If $y = x^2$ is written as $y = f(x)$ then $y = (2x)^2$ becomes $y = f(2x)$ and $y = (2(x + 1))^2$ becomes $y = f(2(x + 1))$.
In $y = f(2x)$ the curve for $y = f(x)$ has stretched by a factor of $\frac{1}{2}$ unit from the y-axis.
In $y = f(2(x + 1))$ the curve for $y = f(x)$ has been moved 1 unit to the left and then been stretched by a factor of $\frac{1}{2}$ unit from the y-axis.

MAKING CONNECTIONS

Dilation from the *y*-axis of the graph of a function

Move the sliders to explore the effects of changing a and b on the graph of a function $f(a(x + b))$.

Summary—transformations of graphs

Given $y = f(x)$, then:

- $y = f(x) + c$ translates the curve c units up
- $y = f(x + b)$ translates the curve b units to the left
- $y = kf(x + b)$ stretches (dilates) the curve by a factor of k from the x-axis
- $y = f(a(x + b))$ stretches (dilates) the curve by a factor $\frac{1}{a}$ from the y-axis.

The order in which transformations are performed is important. Applying transformations in a different order may change the shape of the graph.

Example 2

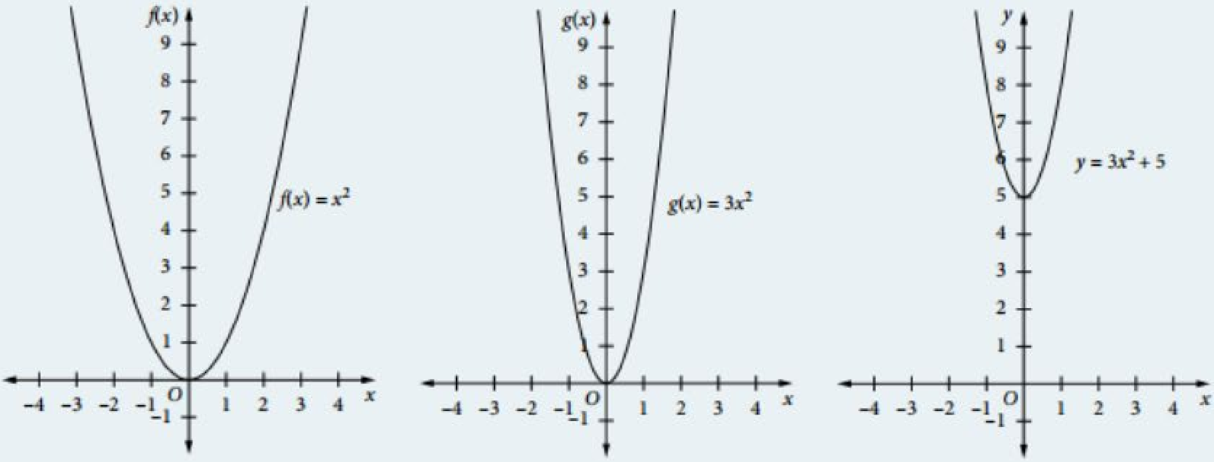
(a) On successive diagrams, draw the graphs of $f(x) = x^2$, $g(x) = 3f(x)$ and $y = g(x) + 5$.

(b) On successive diagrams, draw the graphs of $f(x) = x^2$, $g(x) = f(x) + 5$ and $y = 3g(x)$.

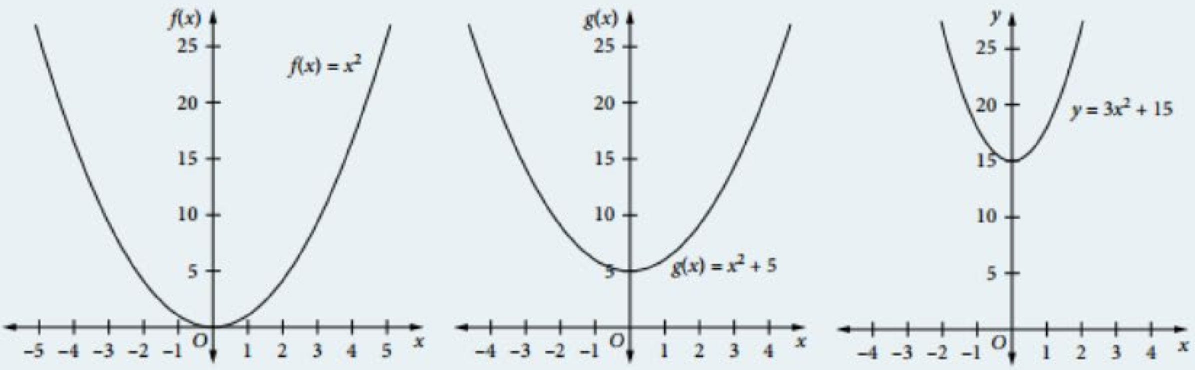
(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

Solution

(a)



(b)



(c) In part **(b)** the vertical translation was 15 instead of 5. The dilation was the same in each case.

The order in which a series of transformations is applied to a function is important.

Reversing or changing the order can change the final function.

EXPLORE FURTHER

The order of transformations

Use technology to explore how the order of transformations changes the graph of the function.

EXERCISE 15.3 TRANSFORMATION OF GRAPHS USING $y = f(ax)$ AND $y = f(a(x + b))$

For the following questions, graphs may be drawn using a table of values or appropriate graphing software.

1 On the same diagram, draw the graph of each equation, stating the dilation and describe any changes to the position of the original graph:

(a) $y=\sqrt{x}, y=\sqrt{2x}, y=\sqrt{2(x-3)}$

(b) $y = \tan x, y = \tan 2x, y = \tan 2\left(x+\frac{\pi}{6}\right)$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

(c) $y=e^{-x}, y=e^{-3x}, y=e^{-3(x-1)}$

(d) $y = \cos x, y = \cos\frac{x}{2}, y = \cos\left(\frac{1}{2}\left(x-\frac{\pi}{4}\right)\right)$ for $0 \le x \le 2\pi$

2 If $f(x) = x^3$, write down the new equation obtained by applying the condition given in each part. Simplify your answer where possible.

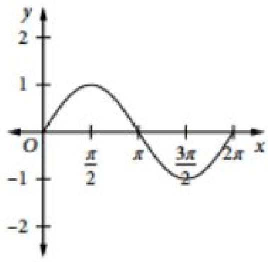
(a) $f(2x)$ **(b)** $f(x-1)$ **(c)** $f(x)+3$ **(d)** $2f(x)+1$ **(e)** $3f(2(x+2))-4$

3 If $f(x) = \cos\frac{x}{2}$, write down the new equation obtained by applying the condition given in each part.

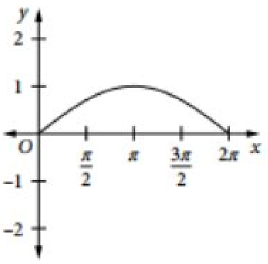
(a) $f(2x)$ **(b)** $f\left(x-\frac{\pi}{3}\right)$ **(c)** $2f(x)$ **(d)** $f(x)-1$ **(e)** $2f\left(x+\frac{\pi}{6}\right)+1$

4 Using graphing software, draw the graphs of the new equation for each part of question **2**.

5 The diagram to the right shows the graph of $y = f(x)$. Which diagram below shows the graph of $y = f\left(\frac{x}{2}\right)$?



A



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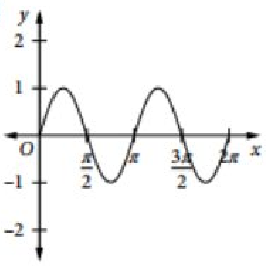

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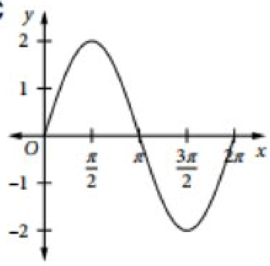
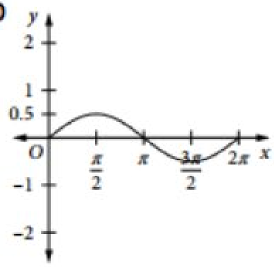


D



6 **(a)** On successive diagrams, draw the graphs of $f(x) = x^2$, $g(x) = 2f(x)$ and $y = g(x) - 3$.

(b) On successive diagrams, draw the graphs of $f(x) = x^2$, $g(x) = f(x) - 3$ and $y = 2g(x)$.

(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

7 **(a)** On successive diagrams, draw the graphs of $f(x) = x + 1$, $g(x) = [f(x)]^2$ and $y = g(x) + 2$.

(b) On successive diagrams, draw the graphs of $f(x) = x + 1$, $g(x) = f(x) + 2$ and $y = [g(x)]^2$.

(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

8 **(a)** On successive diagrams, draw the graphs of $f(x) = x^3$, $g(x) = f(2x)$ and $y = g(x) + 1$.

(b) On successive diagrams, draw the graphs of $f(x) = x^3$, $g(x) = f(x) + 1$ and $y = g(2x)$.

(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

9 **(a)** On successive diagrams, draw the graphs of $f(x) = \sin x$, $g(x) = f(2x)$ and $y = g(x) - 1$ for $0 \le x \le 2\pi$.

(b) On successive diagrams, draw the graphs of $f(x) = \sin x$, $g(x) = f(x) - 1$ and $y = g(2x)$.

(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

10 **(a)** On successive diagrams, draw the graphs of $f(x) = e^x$, $g(x) = f(x - 1)$ and $y = 2g(x)$.

(b) On successive diagrams, draw the graphs of $f(x) = e^x$, $g(x) = 2f(x)$ and $y = g(x - 1)$.

(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

15.4 GRAPHING RATIONAL ALGEBRAIC FUNCTIONS

Functions with the independent variable in the denominator generate curves that are not continuous and may have asymptotes. They may not have any turning points. You need to consider what happens to the function for very large positive and negative values of the variable.

Example 3

Sketch the graph of $y = \frac{1}{x-2}$.

Solution

Because $x - 2 \neq 0$, the function is not defined for $x = 2$, so at $x = 2$ there is a vertical asymptote.

For $x > 2$, $x - 2 > 0$, so $y > 0$. As $x \to 2$ from above, $x - 2$ is a very small positive number and so $y \to \infty$.
This can be written as: $x \to 2^+$, $y \to \infty$.

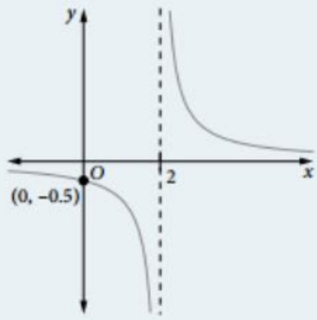
For $x < 2$, $x - 2 < 0$, so $y < 0$. As $x \to 2$ from below, $x - 2$ is a very small negative number and so $y \to -\infty$.
This can be written as: $x \to 2^-$, $y \to -\infty$.

The numerator of $y = \frac{1}{x-2}$ is never zero, so the curve does not cut the x-axis.

For $x = 0$, $y = -0.5$, so the y-intercept is -0.5.

As $x \to \infty$, $y \to 0$ from above; as $x \to -\infty$, $y \to 0$ from below. Thus $y = 0$ is a horizontal asymptote.

For stationary points, find $\frac{dy}{dx}$: $\quad \frac{dy}{dx} = \frac{-1}{(x-2)^2}$, $x \neq 2$



Hence $\frac{dy}{dx} < 0$ for all x in the domain, because $(x - 2)^2 > 0$ in the domain.

Thus $y = \frac{1}{x-2}$ is a decreasing function in each part of its domain.

Also $\frac{dy}{dx} \neq 0$ in the domain, so there are no turning points.

For $x > 2$, $y > 0$; for $x < 2$, $y < 0$.

The curve is concave down for $x < 2$ and concave up for $x > 2$.

Example 4

Sketch the graph of $y = x + \frac{1}{x}$. For what values of x is the curve concave up? What is the range of the function?

Solution

$x \neq 0$: does not cut y-axis and $x = 0$ is a vertical asymptote.

$y = 0$: $x + \frac{1}{x} = 0$ or $\frac{x^2+1}{x} = 0$

Because $x^2 + 1 \neq 0$ for real x: does not cut x-axis.

As $x \to \infty$, $y \to x + [\text{very small amount}] \to x + 0$, so $y \to x$ from above.

As $x \to -\infty$, $y \to x - [\text{very small amount}] \to x - 0$, so $y \to x$ from below.

$\therefore$ $y = x$ is a sloping asymptote.

For stationary points, find $\frac{dy}{dx}$: $\frac{dy}{dx} = 1 - \frac{1}{x^2}, x \neq 0$

Hence for $\frac{dy}{dx} = 0$: $\frac{x^2-1}{x^2} = 0$, so $x = -1, 1$ (for which $y = -2, 2$)

$\therefore$ stationary points at $(-1,-2)$ and $(1,2)$.

Second derivative: $\frac{d^2y}{dx^2} = 0 - \frac{-2}{x^3} = \frac{2}{x^3}$

At $(-1,-2)$: $\frac{d^2y}{dx^2} = -2 < 0$

$\therefore$ $(-1,-2)$ is a local maximum turning point.

At $(1,2)$: $\frac{d^2y}{dx^2} = 2 > 0$

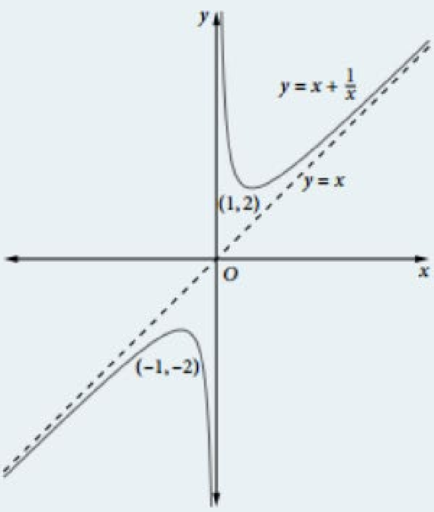
$\therefore$ $(1,2)$ is a local minimum turning point.

Because $\frac{d^2y}{dx^2} = \frac{2}{x^3}$ and $x \neq 0$, there are no points of inflection.

The curve is concave up for $x > 0$.

The range of the function is real $y, |y| \geq 2$.

Consider: why is the curve between the lines $x = 0$ and $y = x$?



Summary—rational algebraic function graphs

When sketching rational algebraic functions:

- identify any restrictions on the domain and the range
- find the intercepts on the coordinate axes where possible (it is usually easy to find the y-intercept, if it exists, but it is not always possible to find the x-intercept)
- use the symmetry properties of odd and even functions whenever possible
- find stationary points and determine their nature
- find asymptotes and use them to guide the shape of the curve. Asymptotes may be horizontal, vertical, sloping or occasionally another curve.

Remember that the shape of your graph sketches can be verified using graphing software.

EXERCISE 15.4 GRAPHING RATIONAL ALGEBRAIC FUNCTIONS

1 The asymptotes of $y = \frac{1}{x+2}$ are:

A $y = 0$ and $x = -2$ **B** $y = 0$ and $x = 2$ **C** $x = 0$ and $y = -2$ **D** $x = 0$ and $y = 2$

2 Sketch the graph of each function. For what values of x is the curve concave down? State the range of each function.

(a) $y = \frac{1}{x+2}$ **(b)** $y = \frac{1}{x-1}$ **(c)** $y = \frac{1}{2-x}$

3 **(a)** Show that the function $y = \frac{x-1}{x-2}$ can be written as $y = 1 + \frac{1}{x-2}$.

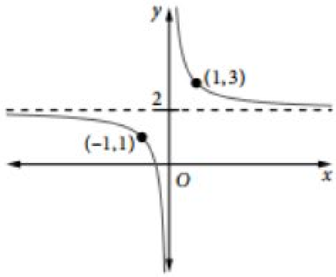
(b) Hence sketch the graph of $y = \frac{x-1}{x-2}$, showing all the asymptotes.

4 Sketch the graph of each function, showing all turning points and points of inflection. For what values of x is the curve concave up? State the range of each function.

(a) $y = x + \frac{4}{x}$ **(b)** $y = x - \frac{1}{x}$ **(c)** $y = 2x + \frac{8}{x}$

5 For the function given in the sketch, state whether each statement below is correct or incorrect.

(a) The horizontal asymptote is $y = 2$.
(b) The curve is continuous.
(c) The curve is concave up for $x > 0$.
(d) The equation of the function is $y = 2 + \frac{1}{x}$.



6 Sketch the graph of each function. For what values of x is the curve concave down? State the range of each function.

(a) $y = 1 + \frac{1}{x+2}$ **(b)** $y = \frac{x-1+1}{x-1} = 1 + \frac{1}{x-1}$ **(c)** $y = \frac{x-2}{x-3}$

7 Sketch the graph of each function, showing all turning points and points of inflection. State the range of each function.

(a) $y = x + \frac{4}{x-1}$ **(b)** $y = x + 2 + \frac{4}{x-1}$ **(c)** $y = x + 3 + \frac{1}{x-7}$

(d) $y = \frac{1}{2x-3}$ **(e)** $y = \frac{x}{4} + \frac{1}{x}$ **(f)** $y = |x| + \frac{1}{x}$

15.5 APPLICATIONS INVOLVING GRAPHING FUNCTIONS

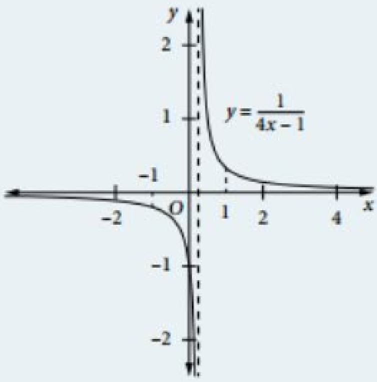
Graphs of functions with algebraic denominators may have asymptotes. You also have information about turning points and points of inflection obtained by using calculus to differentiate the function. You can use all these skills to sketch a variety of functions.

Example 5

(a) Sketch the graph of $y = \frac{1}{4x-1}$.
(b) Find the equation of the tangent to the curve at the point where $x = 1$.
(c) Find the equation of the normal to the curve at point where $x = -1$.
(d) Find the coordinates of the point of intersection of the tangent and normal found in parts **(b)** and **(c)**.

Solution

(a)



(b) $y=\frac{1}{4x-1}$: $\frac{dy}{dx}=-\frac{4}{(4x-1)^2}$

$x=1$: Gradient of tangent $=-\frac{4}{9}$

$x=1, y=\frac{1}{3}$

Equation of tangent: $y-\frac{1}{3}=-\frac{4}{9}(x-1)$

$$9y-3=-4x+4$$

$$4x+9y-7=0$$

(c) $x=-1$: Gradient of tangent $=-\frac{4}{25}$

Gradient of normal $=\frac{25}{4}$

$x=-1, y=-\frac{1}{5}$

Equation of normal: $y+\frac{1}{5}=\frac{25}{4}(x+1)$

$$20y+4=125x+125$$

$$125x-20y+121=0$$

(d) $4x+9y-7=0$ [1]

$125x-20y+121=0$ [2]

$[1]\times 20$: $80x+180y-140=0$ [3]

$[2]\times 9$: $1125x-180y+1089=0$ [4]

$[3]+[4]$: $1205x+949=0$

$$x=-\frac{949}{1205}$$

Substitute into [1]: $-\frac{3796}{1205}+9y-7=0$

$$9y=\frac{12231}{1205}$$

$$y=\frac{1359}{1205}$$

The point of intersection is $\left(-\frac{949}{1205},\frac{1359}{1205}\right)$

EXERCISE 15.5 APPLICATIONS INVOLVING GRAPHING FUNCTIONS

1 **(a)** Sketch the graph of $y=\frac{1}{2x-1}$.
(b) Find the equation of the tangent to the curve at the point where $x=1$.
(c) Find the equation of the normal to the curve at point where $x=-1$.
(d) Find the coordinates of the point of intersection of the tangent and normal found in parts **(b)** and **(c)**.

2 **(a)** Sketch the curve $y=x+\frac{1}{x}$, showing its asymptotes.
(b) Find the coordinates of the turning points of $y=x+\frac{1}{x}$ and determine their nature.
(c) What is the least value of $x+\frac{1}{x}$ over the domain $x>0$?

3 **(a)** Sketch the graph of $f(x)=e^x+4e^{-x}$.
(b) For what values of x is $f'(x)>0$?
(c) What is the minimum value of $f(x)$ and when does it occur?

4 **(a)** Sketch the graph of $f(t)=\frac{5}{2+3e^{-t}}$, $t\geq 0$.
(b) Show that $f'(t)>0$ for all values of t in the domain.
(c) Find $\lim_{t\to\infty} f(t)$.
(d) What is the range of the function?

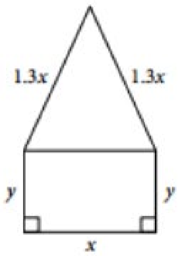
5 $f(x)$ is defined by the rule $f(x)=e^{-x}\cos x$ over the domain $-\frac{\pi}{2}\leq x\leq\frac{3\pi}{2}$.
(a) Find the values of $f(0)$, $f\left(\frac{\pi}{2}\right)$, $f(\pi)$.
(b) Find $f'(x)$.
(c) Show that $f'(0)=-1$, $f'\left(\frac{3\pi}{4}\right)=0$ and $f'\left(-\frac{\pi}{4}\right)=0$.
(d) Sketch the graph of $y=f(x)$.
(e) Find the maximum value of $f(x)$ over the domain and the value of x for which it occurs.

6 **(a)** For what values of x is $f(x) = \log_e (\sin x)$ defined over the domain $0 \le x \le 2\pi$?
(b) Find $f'(x)$ and state its domain. **(c)** Find the maximum value of $f(x)$ over its domain.
(d) Sketch the graph of $y = f(x)$ over its domain.

7 **(a)** If $y = \log_e (1 + \sin x)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **(b)** For what values of x is $\frac{dy}{dx} = 0$ over $0 \le x \le 4\pi$?
(c) When is the function a maximum? **(d)** Explain why the function has no points of inflection.
(e) Sketch $y = \log_e (1 + \sin x)$ for $0 \le x \le 4\pi$.

8 The concentration C of insects per square metre in a forest is given by
$C(t) = 1000\left[\cos\left(\frac{\pi(t-8)}{2}\right)+2\right]^2 - 1000$, for $8 \le t \le 16$, where t is the number of hours after midnight.
(a) What is the minimum concentration of insects and when does it occur?
(b) Sketch the function for the given domain.

9 The diagram consists of a rectangle surmounted by an isosceles triangle with dimensions as shown.



(a) Show that the height of the isosceles triangle is $1.2\,x$.
(b) Show that the total area of the figure is given by $A = xy + 0.6x^2$.
(c) If the perimeter of the figure is 48 metres, express y in terms of x.
(d) Find the expression for $A(x)$ as a function of x only.
(e) Sketch the graph of $y = A(x)$.
(f) Find the dimensions of the diagram that give a maximum area and state that area.

10 The current in an electrical circuit t seconds after the power is switched on is given by
$I(t) = 100(1 - e^{-5t})$ amps, $t \ge 0$.
(a) Find the current when $t = 0$, 0.2 and 1 second.
(b) As t increases, describe what happens to the current. **(c)** Find $I'(t)$ and draw the graph of $y = I'(t)$.
(d) Draw the graph of $y = I(t)$. **(e)** What does the graph in part **(c)** represent on the graph in part **(d)**?

15.6 GRAPHICAL SOLUTION OF EQUATIONS

There are usually standard techniques for solving equations associated with various functions, e.g. linear equations, quadratic equations and trigonometric equations. However, there is usually no standard technique for solving equations that combine two or more different functions, for example:

- $\sin x = e^x$
- $\cos x = \log x$

Equations like this 'transcend' (go beyond) algebraic equation-solving techniques and hence are called **transcendental equations**. You must instead use various non-algebraic methods to find approximate solutions. The most accurate method is to calculate numerical solutions using graphing software. Another method is to sketch the graphs of the two functions, use the graphs to find approximate x-values of any intersection points, then use a calculator to refine these approximate x values as closely as possible. This section extends the scope of the equations covered in Section 12.3.

EXPLORE FURTHER

Graphical solutions of equations

Use technology to solve transcendental equations graphically.

EXERCISE 15.6 GRAPHICAL SOLUTION OF EQUATIONS

1 By drawing graphs of the given functions, determine how many solutions exist for the given equation.

(a) $y = x^2, y = 2x - 1$
Equation: $x^2 - 2x + 1 = 0$

(b) $y = x^2, y = 3x + 1$
Equation: $x^2 - 3x - 1 = 0$

(c) $y = x^2, y = x - 4$
Equation: $x^2 - x + 4 = 0$

(d) $y = x^3, y = 2x$
Equation: $x^3 - 2x = 0$

(e) $y = x^3 - x, y = x^2$
Equation: $x^3 - x^2 - x = 0$

(f) $y = e^x, y = x + 2$
Equation: $e^x - x - 2 = 0$

2 By drawing graphs of the given functions for the given domain, determine how many solutions exist for the given equation.

(a) $y = \sin x, y = \frac{x}{2}$
Domain: $-2\pi \le x \le 2\pi$
Equation: $\sin x - \frac{x}{2} = 0$

(b) $y = \log_e x, y = x - 2$
Domain: $0 \le x \le 2\pi$
Equation: $\log_e x - x + 2 = 0$

(c) $y = 2\cos x, y = \log_e x.$
Domain: $0 \le x \le 2\pi$
Equation: $2\cos x - \log_e x = 0$

(d) $y = e^x, y = \sin x$
Domain: $-2\pi < x < 2\pi$
Equation: $e^x - \sin x = 0$

(e) $y = e^{-x}, y = \sin x$
Domain: $-2\pi \le x \le 2\pi$
Equation: $e^{-x} - \sin x = 0$

(f) $y = e^{-x}, y = \tan x$
Domain: $-\frac{\pi}{2} < x < \frac{3\pi}{2}$
Equation: $e^{-x} - \tan x = 0.$

3 Show graphically that the equation $8\log_{10}(0.1x + 0.5) = 2 - x$ has a solution between $x = 2$ and $x = 4$. Find this solution correct to 2 decimal places.

15.7 REGIONS AND INEQUALITIES

A straight line represents a function (unless it is a vertical line). If the equation of a straight line is changed to an inequality, then the function becomes a relation. It is no longer a straight line, but instead it can be represented graphically by a region in the number plane.

To graph a region on the number plane:

- Graph the equation of the region's boundary.
- Select a point not on the boundary.
- Substitute the coordinates of this point into the equation of the boundary.
- If the point makes the inequality true, then all points on that side of the inequality will also make it true. Shade that side of the boundary to indicate the region. (If the point does not make the inequality true, then the points on the other side of the inequality must, so shade that side instead.)

If the inequality includes '... or equal to', then the boundary is part of the region. If the inequality does not include '... or equal to', then the boundary is not part of the region, so it should be dashed to show this.

Example 6

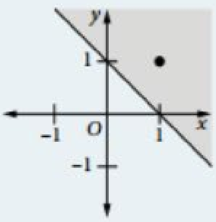
Graph the region in the number plane represented by each inequality.

(a) $x + y \ge 1$ **(b)** $x + y < 1$ **(c)** $x + y \le 1$ **(d)** $x + y > 1$

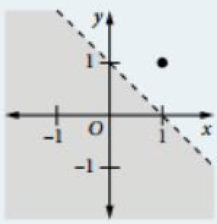
Solution

In each case, first draw the line $x + y = 1$. In parts **(b)** and **(d)**, dash the line to show that the boundary is not part of the region. Substitute the non-boundary point $(1, 1)$ into each inequality to find the region side.

(a) $x + y \ge 1$
Solid line
$2 > 1$
$\text{LHS} = 1 + 1 = 2 > 1$
Result true:
shade above the line



(b) $x + y < 1$
Dashed line
$2 > 1$
$\text{LHS} = 1 + 1 = 2 > 1$
Result not true:
shade below the line



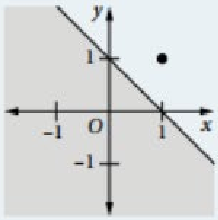
(c) $x + y \le 1$

Solid line

$2 > 1$

$\text{LHS} = 1 + 1 = 2 > 1$

Result not true:
shade below the line



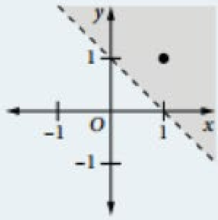
(d) $x + y > 1$

Dashed line

$2 > 1$

$\text{LHS} = 1 + 1 = 2 > 1$

Result true:
shade above the line



The regions in **(a)** and **(b)** together make the whole number plane, as do the regions in **(c)** and **(d)** together.

The boundary line divides the number plane into three sets of points: the points on the line, the points above the line and the points below the line. (If the inequality includes '… or equal to', then the boundary is part of the region; if the inequality does not include '… or equal to', then the boundary is not part of the region, so it is dashed.)

Thus for the example above, the region in part **(a)** could be described as 'the set of points on or above the line with equation $x + y = 1$'. The region in part **(b)** could be described as 'the set of points below the line with equation $x + y = 1$'.

Example 7

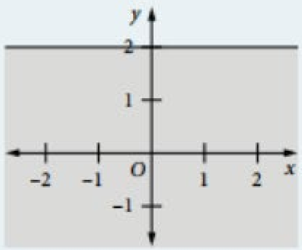
Graph the region in the number plane represented by each inequality.

(a) $y \le 2$ **(b)** $-1 < y \le 2$ **(c)** $x > 1$ **(d)** $x > 1$ or $x \le -1$

Solution

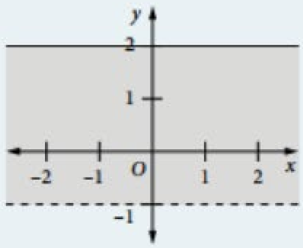
These inequalities have only one variable, so their boundaries are either horizontal lines (as in **(a)** and **(b)**) or vertical lines (as in **(c)** and **(d)**).

(a) $y \le 2$



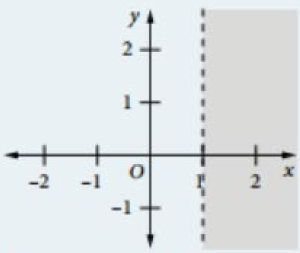
Region is below the line

(b) $-1 < y \le 2$



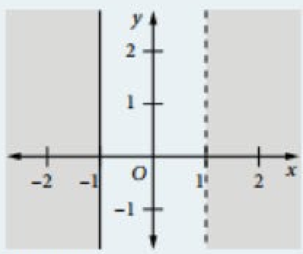
Region is between the lines

(c) $x > 1$



Region is to the right of the line

(d) $x > 1$ or $x \le -1$



Region is outside the lines

Example 8

(a) Sketch the region defined by the intersection $y \geq x^2 - 1$ and $y \leq 3x + 3$.

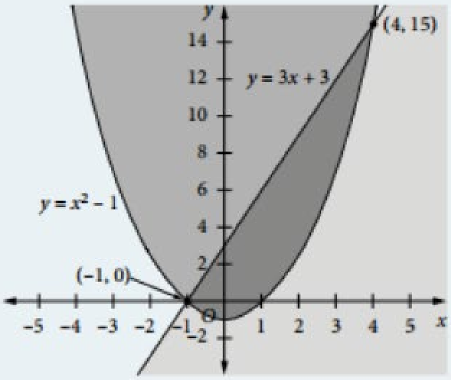
(b) Hence write the solution to $x^2 - 3x - 4 \leq 0$.

(c) Solve $x^2 - 3x - 4 \leq 0$ algebraically to check your solution to **(b)**.

(d) What would be different in this process if you were solving $x^2 - 3x - 4 < 0$?

Solution

(a)



(b) $x^2 - 3x - 4 \leq 0$

Rewrite in terms of given equations: $x^2 - 1 \leq 3x + 3$

Find the points of intersection of the graphs by using simultaneous equations, setting each rule equal to the other: $(-1, 0)$ and $(4, 15)$

Roots of $x^2 - 1 = 3x + 3$ are: $x = -1, 4$

Required region is between the curves: Solution is $-1 \leq x \leq 4$

(c) $x^2 - 3x - 4 \leq 0$

Factorise: $(x + 1)(x - 4) \leq 0$

The roots of $x^2 - 3x - 4 = 0$ are: $x = -1, 4$

Pick a value of x between -1 and 4, e.g. $x = 0$ and substitute into the quadratic expression.

To test $x^2 - 3x - 4 \leq 0$ use $x = 0$.

$x = 0$: $0^2 - 3(0) - 4 \leq 0$

$-4 \leq 0$

Since this value makes the inequality true, it must lie in the region defined where $-1 \leq x \leq 4$.

(d) Graphically, the boundary would not be included so the parabola and the lines would be dashed. The solution would not include equality, it would be $-1 < x < 4$.

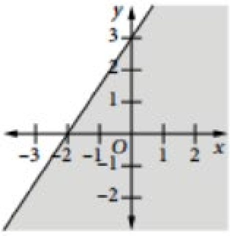
EXERCISE 15.7 REGIONS AND INEQUALITIES

Shade the region represented by each inequality.

1 $y > 2x$	**2** $y < x + 1$	**3** $x \leq 3$	**4** $y \geq 1$
5 $y \leq x + 2$	**6** $2x + 3y \geq 6$	**7** $2x + y < 1$	**8** $0 \leq x \leq 2$
9 $-2 < y \leq 3$	**10** $3x - 4y \leq 6$	**11** $x - y < -2$	**12** $2y - 5x < 10$
13 $x \leq 1.5$	**14** $\frac{x}{2} + \frac{y}{3} < 1$	**15** $-3 < x + y < 2$	**16** $0 \leq x - y < 3$

17 Which inequality defines the shaded region?

A $3x - 2y + 6 \geq 0$
B $3x - 2y + 6 \leq 0$
C $3x + 2y - 6 \geq 0$
D $3x + 2y - 6 \leq 0$



18 Shade the region in the number plane that:

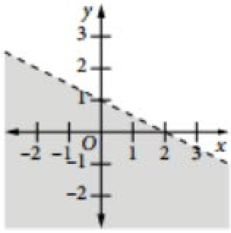
(a) is above the line $y = 3x + 2$
(b) is on or below the line $x - 2y + 4 = 0$
(c) is above the line $y = -1$
(d) is on or to the left of the line $x = 3$
(e) is between the lines $2x + 3y + 6 = 0$ and $2x + 3y - 6 = 0$
(f) is on or above the x-axis.

19 Describe in words the regions defined by the following inequalities:

(a) $y < x + 2$ **(b)** $y \geq x$ **(c)** $x > 3$ **(d)** $y \leq 4$ **(e)** $x + 3y \leq 9$ **(f)** $-2 \leq x < 3$

20 For this graph, indicate whether each statement is correct or incorrect.

(a) The equation of the boundary is $x + 2y - 2 = 0$.
(b) The gradient of the boundary line is $\frac{1}{2}$.
(c) The inequality for the region is $x + 2y - 2 > 0$.
(d) The inequality for the region is $x + 2y - 2 < 0$.



21 **(a)** Sketch the region defined by the intersection $y \geq x^2 - 1$ and $y \leq 3 - 3x$.
(b) Hence write the solution to $x^2 + 3x - 4 \leq 0$.

22 **(a)** Sketch the region defined by the intersection $y \leq 3 - x^2$ and $y \geq 2x$.
(b) Hence write the solution to $x^2 + 2x - 3 \leq 0$.

23 **(a)** Sketch the region defined by the intersection $y \geq x^2 + x$ and $y \leq 2x + 2$.
(b) Hence write the solution to $x^2 - x - 2 \leq 0$.

24 **(a)** Sketch the region defined by the intersection $y < 2x - x^2$ and $y \geq 3x - 2$.
(b) Hence write the solution to $x^2 + x - 2 < 0$.
(c) Solve $x^2 + x - 2 > 0$ algebraically to check your solution to **(b)**.

25 **(a)** Sketch the region defined by the intersection $y \geq x^2$ and $y \leq x + 3$.
(b) Hence write the solution to $x^2 - x - 3 \leq 0$.
(c) Solve $x^2 - x - 3 \leq 0$ algebraically to find the exact solution to **(b)**.

15.8 SIMULTANEOUS LINEAR INEQUALITIES

Two linear equations may intersect at a point. The intersection of two linear inequalities is the region common to the two inequalities. This is the region where both inequalities hold simultaneously.

Example 9

(a) Sketch the region given by $y \geq x$. **(b)** Sketch the region given by $x + y < 2$.

(c) Sketch the region common to $y \geq x$ and $x + y < 2$.

Solution

(a) $y \geq x$

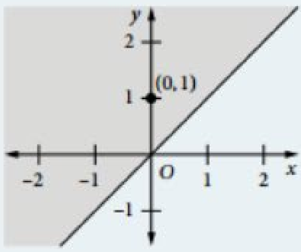
Test the point $(0, 1)$

Test: LHS $\geq$ RHS using (0,1)

$y > -x$, sub in $y = 1$ and $x = 0$

$1 >= 0$, which is true.

Shade region above line



(b) $x + y < 2$

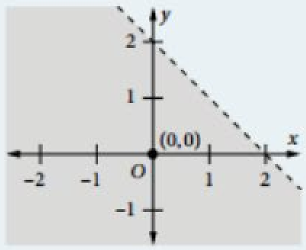
Test the point $(0, 0)$

Test LHS < RHS using the point (0,0)

$x + y < 2$, where $x = 0$ and $y = 0$

$0 + 0 < 2$, which is true.

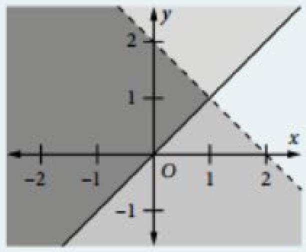
Shade region below dashed line



(c) $y \geq x$ and $x + y < 2$

Identify common region

Shade clearly, using a darker shading for the common region



Example 10

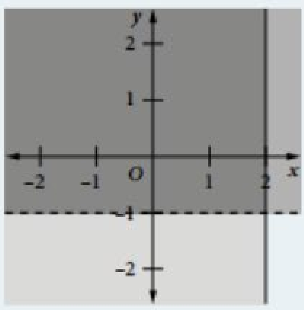
Sketch the region defined by each pair of inequalities. Describe the region in words.

(a) $x \leq 2, y > -1$ **(b)** $y \leq x + 1, x \leq 1$

Solution

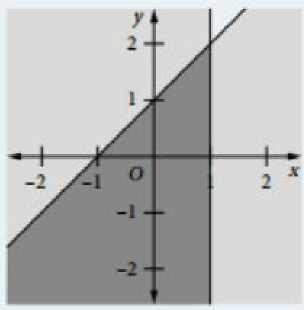
Draw each boundary and shade the two regions, then shade the common region differently.

(a) $x \leq 2, y > -1$



The region on and to the left of the line $x = 2$ that is also above the line $y = -1$.

(b) $y \leq x + 1, x \leq 1$



The region on and below the line $y = x + 1$ that is also on and to the left of the line $x = 1$.

If the shading of different regions becomes difficult to show, you should just lightly shade the original regions before darkening the final answer, as in part **(b)**.

Example 11

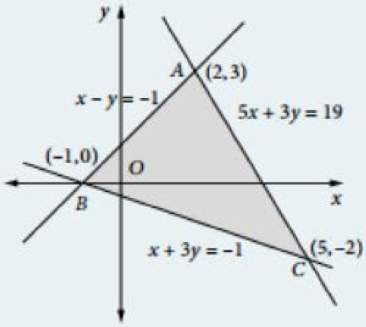
Sketch the region defined by the three inequalities $x - y \geq -1$, $x + 3y \geq -1$, $5x + 3y \leq 19$. Show the points of intersection of the lines. Describe the region in words.

Solution

To find the points of intersection of the lines, solve pairs of equations simultaneously:

- The lines $x - y = -1$ and $5x + 3y = 19$ intersect at $A(2, 3)$
- The lines $x - y = -1$ and $x + 3y = -1$ intersect at $B(-1, 0)$
- The lines $5x + 3y = 19$ and $x + 3y = -1$ intersect at $C(5, -2)$

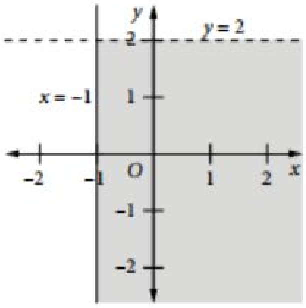
The shaded region is the interior of the triangle bounded by the lines $x - y = -1$, $x + 3y = -1$ and $5x + 3y = 19$. The vertices of the region are the intersection points A, B and C.



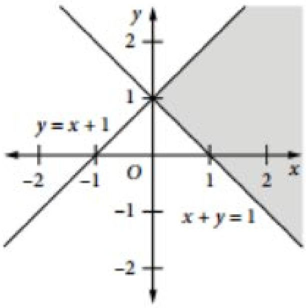
EXERCISE 15.8 SIMULTANEOUS LINEAR INEQUALITIES

1 Describe the shaded region in each diagram using both words and inequalities.

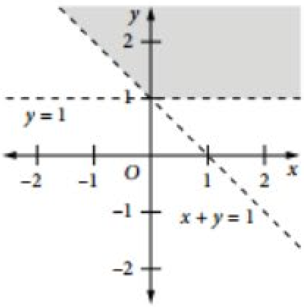
(a)



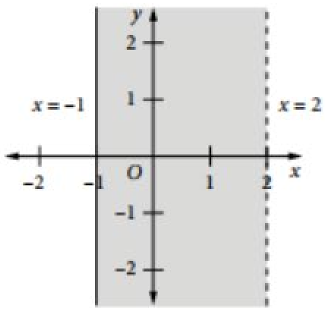
(b)



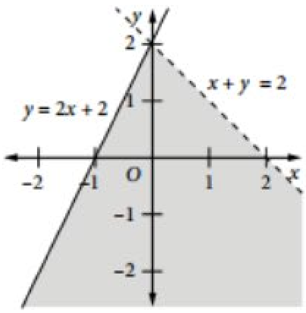
(c)



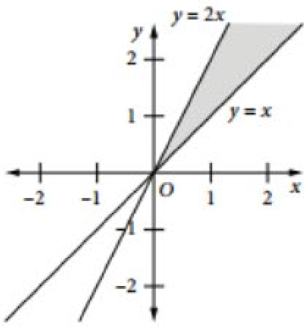
(d)



(e)

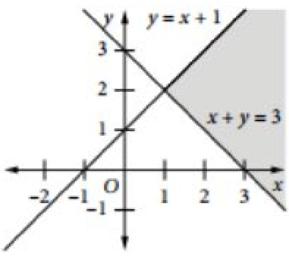


(f)



2 Which of the following points is in the shaded region?

A $(1, 3)$ **B** $(1, 1)$
C $(3, 1)$ **D** $(-1, 3)$



3 What are the inequalities that define the region graphed in question **2**?

4 Graph the regions defined by each set of inequalities. State whether each of the given points is in the region.

(a) $x+y\le 3, y\le x$
$(0,0), (2,3), (-1,-2)$

(b) $2y>x+2, x+y>-1$
$(0,0), (0,1), (2,5)$

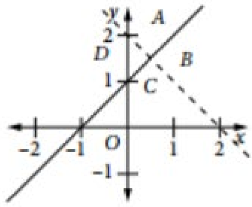
(c) $x+2y\ge 8, y<7$
$(0,4), (-1,1), (9,2)$

(d) $3y\le 2x+6, x+y>2$
$(2,0), (3,3), (4,-1)$

(e) $4x+y\le 4, x\ge -2$
$(0,0), (-3,1), (1,0)$

(f) $y>3x+3, x+y<3$
$(0,3), (2,7), (-1,4)$

5 The two lines divide the number plane into four regions labelled on the diagram A, B, C and D. Indicate whether the inequalities given for each region are correct or incorrect.



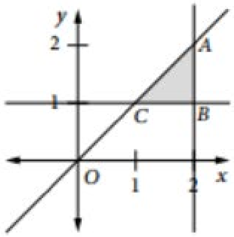
(a) $A: x+y>2, x-y\le -1$ **(b)** $B: x+y>2, x-y\le -1$
(c) $C: x+y<2, x-y\ge -1$ **(d)** $D: x+y<2, x-y\le -1$

6 Shade the region defined by the inequalities. By solving the inequalities simultaneously in pairs, find the coordinates of the vertices of the figure.

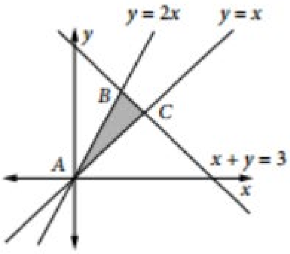
(a) $y\le x+2, 2x+y\le 4, x+y\ge 2$ **(b)** $2y-x\le 4, y>3x-6, 3x+y>-6$
(c) $y-2x<4, y+2x<6, y\ge x+2$ **(d)** $y-3x<3, 3x+4y<12, x-2y<4$
(e) $y\ge x-1, x+y\le 2, x\ge 0, y\ge 0$ **(f)** $x+y\le 2, x+y\ge -2, x-y\le 2, x-y\ge -2$

7 Describe (in words and using inequalities) the regions that are shaded in each part. Find the coordinates of the points A, B and C in each case by solving simultaneously.

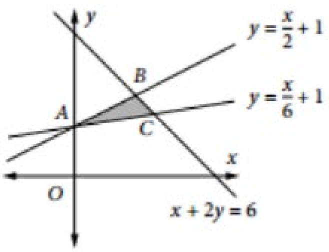
(a)



(b)



(c)



CHAPTER REVIEW 15

1 On the same diagram, draw the graphs of $y=x^3, y=(x+3)^3, y=x^3+3$.

2 On the same diagram, draw the graph of each equation, stating the dilation and other transformations from the first graph: $y=\ln x, y=4\ln x, y=4\ln(x+2)$.

3 On the same diagram, draw the graphs for $0\le x\le 2\pi$:

(a) $y=\sin x, y=\sin\left(x-\frac{\pi}{6}\right), y=\sin\left(x+\frac{\pi}{3}\right)$ **(b)** $y=\sec x, y=\sec x-2, y=\sec x+1$

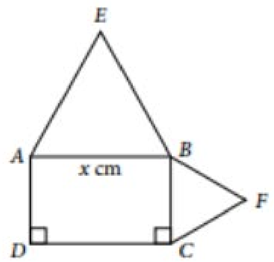
4 If $f(x)=e^x$, write down the new equation obtained by applying the condition given in each part.

(a) $f(2x)$ **(b)** $f(x-3)$ **(c)** $f(x)+1$ **(d)** $2f(x)+4$ **(e)** $3f(2(x+2)-1$

5 Haulage Company makes frequent deliveries from Sydney to Melbourne and calculates that the overhead cost $\$C$ depends on the average delivery speed v km h^{-1} according to the rule $C=v+\frac{6400}{v}$.

(a) What is the domain of this function? **(b)** Sketch the function for this domain.
(c) Find the average delivery speed to minimise the overhead cost.

6 Figure $AEBFCD$ consists of the rectangle $ABCD$ topped by an equilateral triangle AEB on the side AB, with another equilateral triangle BFC on the side BC, as shown in the diagram. The perimeter of the figure is 54 cm.



(a) Obtain a formula for the area of the figure, $A(x)$, in terms of x, the length of the side AB.
(b) What is the domain of this function?
(c) Sketch the function for this domain.
(d) Find the dimensions of this rectangle when the total area is a maximum.

7 **(a)** Given $f(x) = 4 + \frac{3x-1}{x^2}$, find:
(i) the values of x for which $f(x) = 0$ **(ii)** $\lim_{x \to \infty} f(x)$
(iii) the value of $f(x)$ as $x \to 0$ **(iv)** the equation of any asymptotes
(v) the coordinates of the stationary points and determine their nature
(vi) the coordinates of any points of inflection.
(b) Sketch the graph of $y = f(x)$, showing the information obtained from part **(a)**.

8 **(a)** If $y = \frac{4x}{(x-1)^2}$, find:
(i) the coordinates of the stationary points and determine their nature
(ii) the coordinates of any points of inflection **(iii)** the equations of any asymptotes.
(b) Sketch the graph of $f(x) = \frac{4x}{(x-1)^2}$. **(c)** Sketch the graph of $y = f'(x)$.

9 **(a)** Write the expression $\frac{x(x+1)}{x-1}$ in the form $px + q + \frac{r}{x-1}$, where p, q and r are real numbers.
(b) If $y = \frac{x(x+1)}{x-1}$, find the coordinates of the stationary points and determine their nature.
(c) Sketch the graph of $y = \frac{x(x+1)}{x-1}$. **(d)** On your diagram, draw the lines $y = x$ and $y = x + 3$.
(e) For what value of c will the line $y = x + c$ not intersect the graph?

10 Draw the graph of $y = \sin 2x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and use it to solve: **(a)** $\sin 2x = \frac{x}{3}$ **(b)** $\sin 2x = 1 - x$.

11 **(a)** On successive diagrams, draw the graphs of $f(x) = x^2$, $g(x) = f(3x)$ and $y = g(x) - 2$.
(b) On successive diagrams, draw the graphs of $f(x) = x^2$, $g(x) = f(x) - 2$ and $y = g(3x)$.
(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

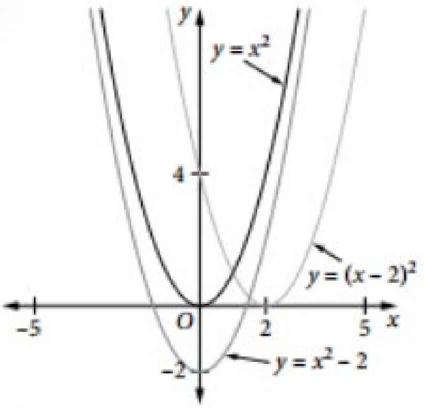
12 **(a)** On successive diagrams, draw the graphs of $f(x) = \cos x$, $g(x) = [f(x)]^2$ and $y = g(x) + 1$ for $0 \le x \le 2\pi$.
(b) On successive diagrams, draw the graphs of $f(x) = \cos x$, $g(x) = f(x) + 1$ and $y = [g(x)]^2$ for $0 \le x \le 2\pi$.
(c) Discuss the differences between your final graphs in parts **(a)** and **(b)**.

13 **(a)** Sketch the region defined by the intersection $y \ge x^2 + 4$ and $y \le 2x + 4$.
(b) Hence write the solution to $x^2 - 2x \le 0$.

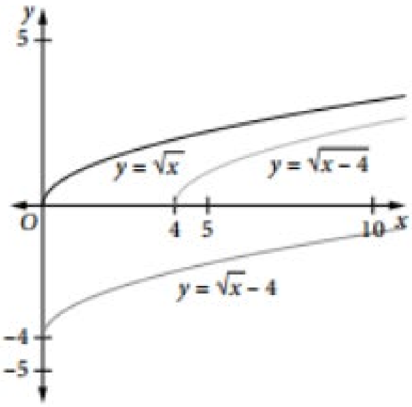
CHAPTER 15

EXERCISE 15.1

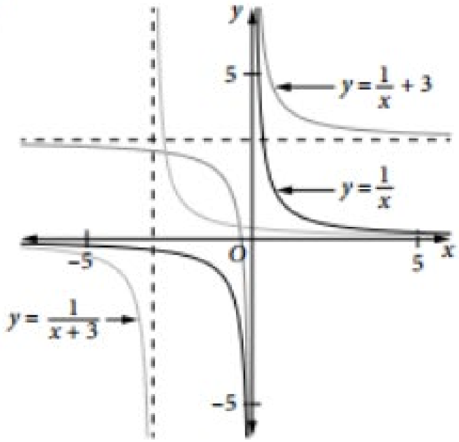
1 (a)



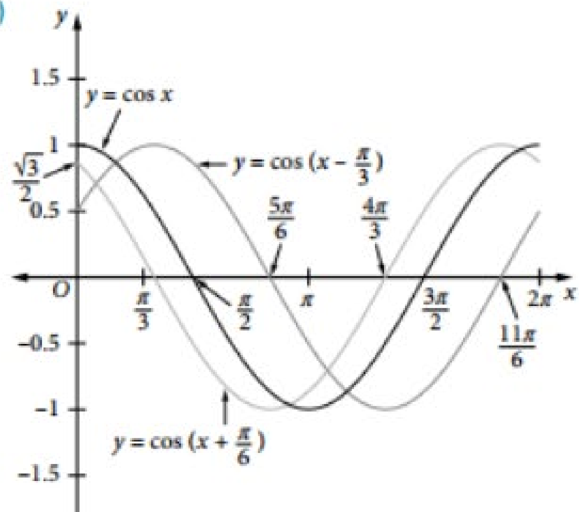
(b)



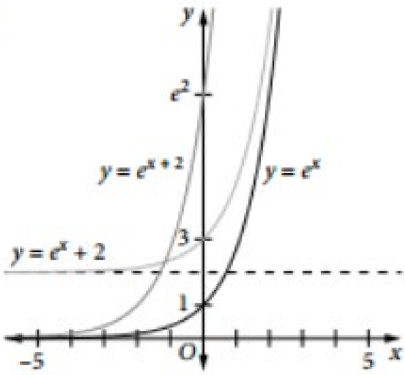
(c)



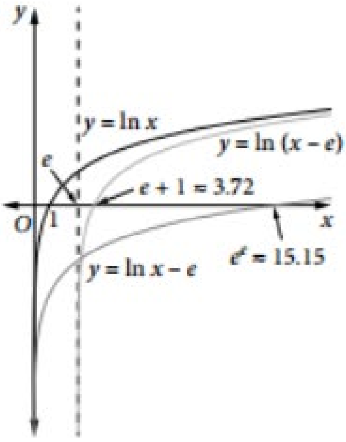
(d)



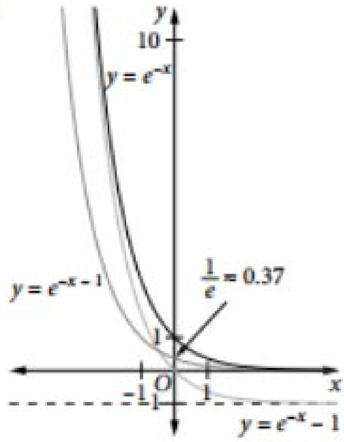
2 (a)



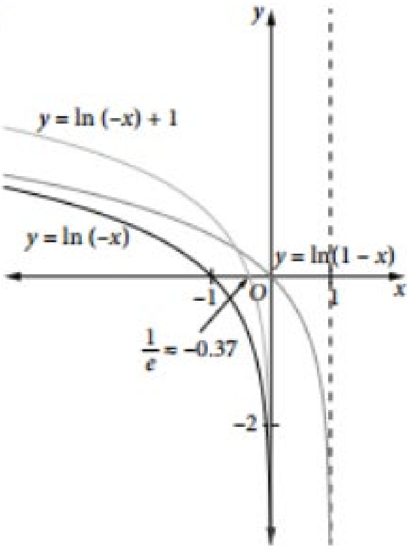
(b)



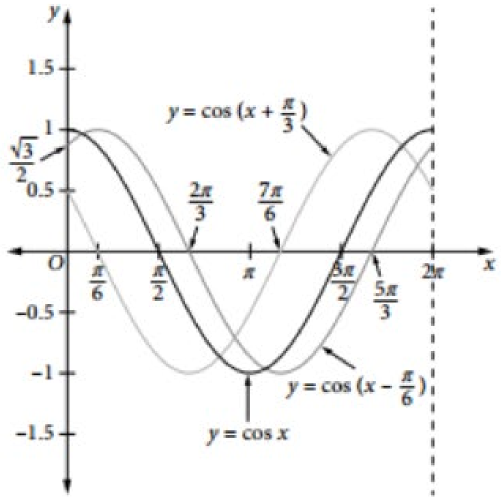
(c)



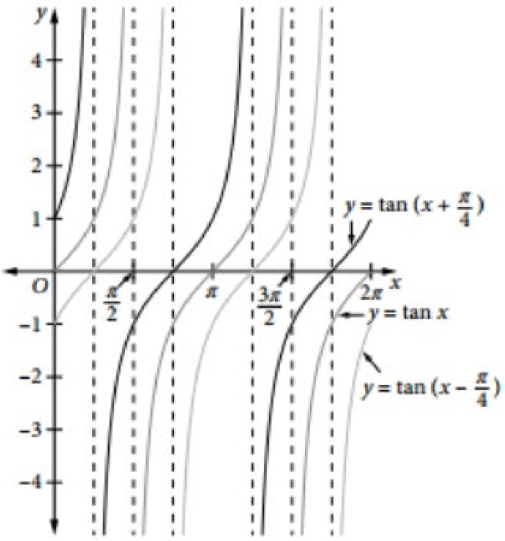
(d)



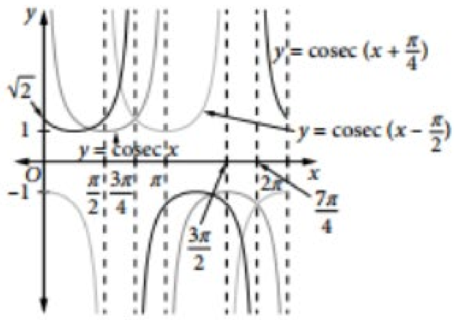
3 (a)



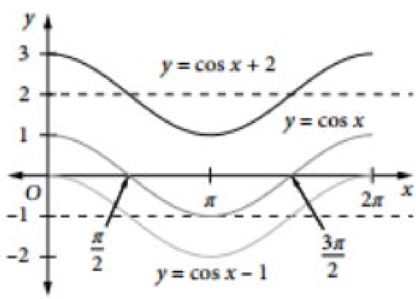
(b)



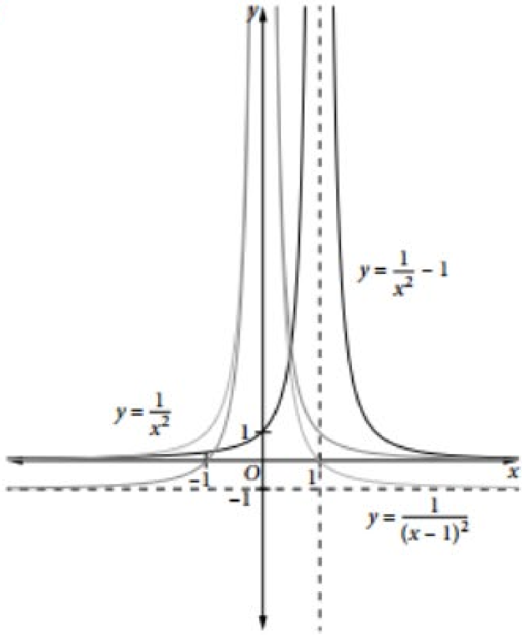
(c)



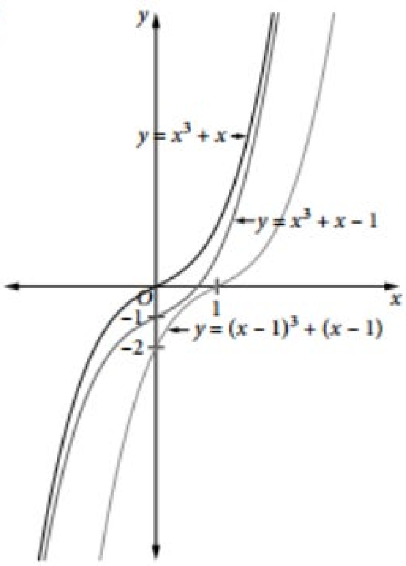
(d)



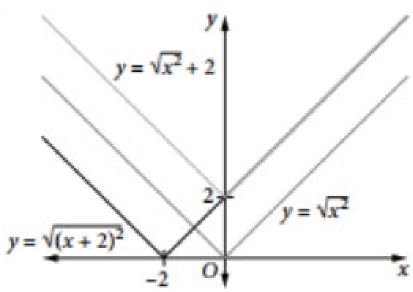
4 (a)



(b)



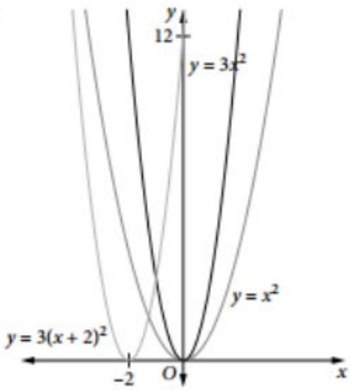
(c)



5 C.

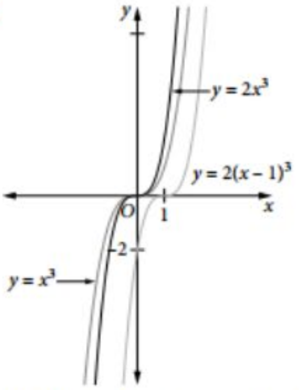
EXERCISE 15.2

1 (a)



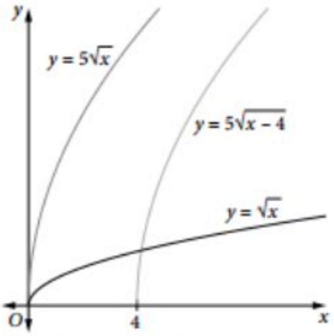
The dilation from the x-axis for the second and third graphs has factor 3.

(b)



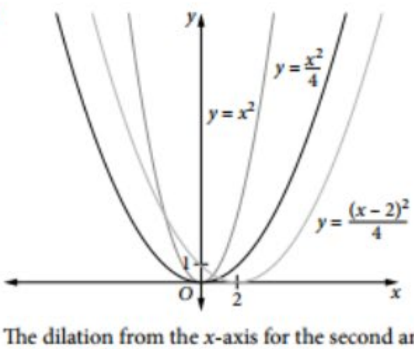
The dilation from the x-axis for the second and third graphs has factor 2.

(c)



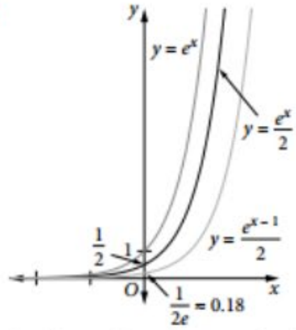
The dilation from the x-axis for the second and third graphs has factor 5.

(d)



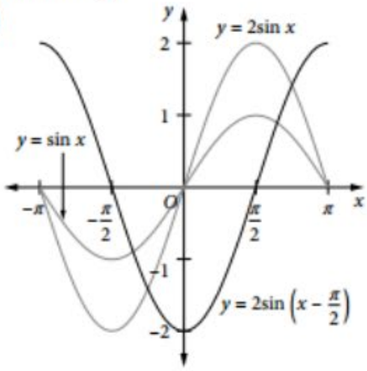
The dilation from the x-axis for the second and third graphs has factor $\frac{1}{4}$.

2 (a)



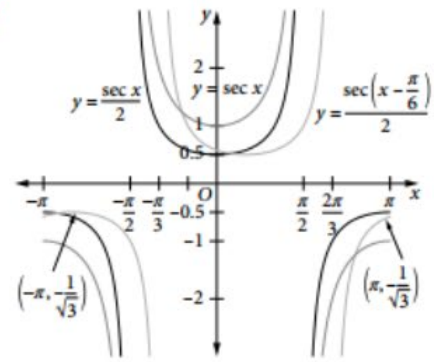
The dilation from the x-axis for the second and third graphs has factor $\frac{1}{2}$.

(b)



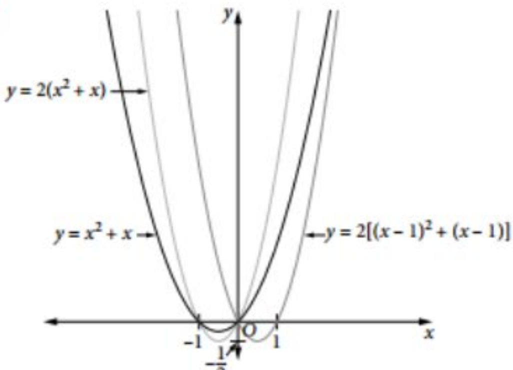
The dilation from the x-axis for the second and third graphs has factor 2.

(c)



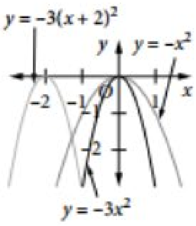
The dilation from the x-axis for the second and third graphs has factor $\frac{1}{2}$.

3 (a)



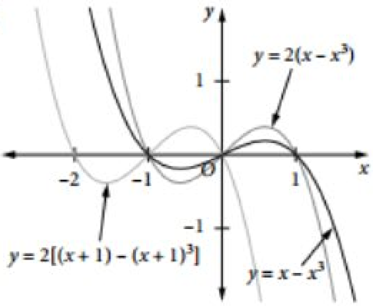
The dilation from the x-axis for the second and third graphs has factor 2.

(b)



The dilation from the x-axis for the second and third graphs has factor 3.

(c)

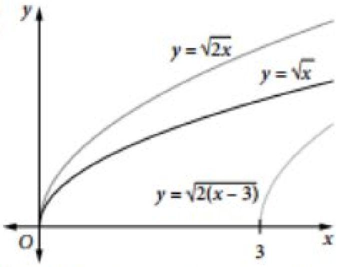


The dilation from the x-axis for the second and third graphs has factor 2.

4 B

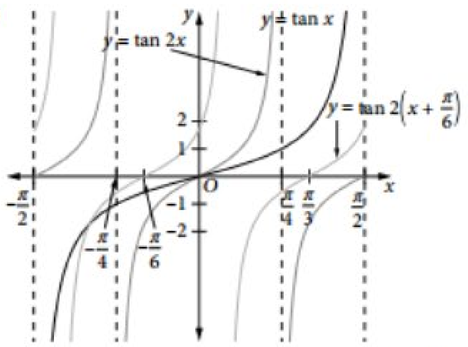
EXERCISE 15.3

1 (a)



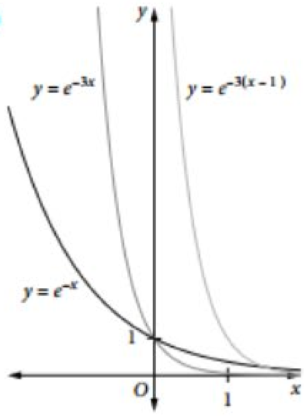
The dilation from the y-axis in the second and third graphs has a factor of 0.5. The third graph has also undergone a translation of 3 units to the right.

(b)



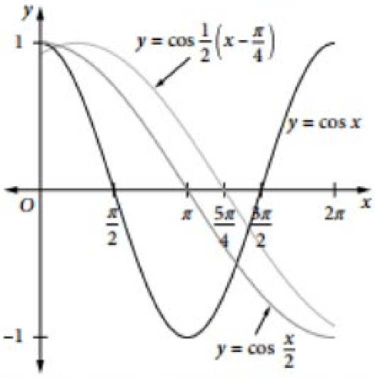
The dilation from the y-axis in the second and third graphs has a factor of 0.5. The third graph has also undergone a translation of $\frac{\pi}{6}$ units to the left.

(c)



The dilation from the y-axis in the second and third graphs has a factor of $\frac{1}{3}$. The third graph has also undergone a translation of 1 unit to the right.

(d)



The dilation from the y-axis in the second and third graphs has a factor of 2. The third graph has also undergone a translation of $\frac{\pi}{4}$ units to the right.

2 (a) $f(2x) = (2x)^3 = 8x^3$ (b) $f(x-1) = (x-1)^3$

(c) $f(x) + 3 = x^3 + 3$ (d) $2f(x) + 1 = 2x^3 + 1$

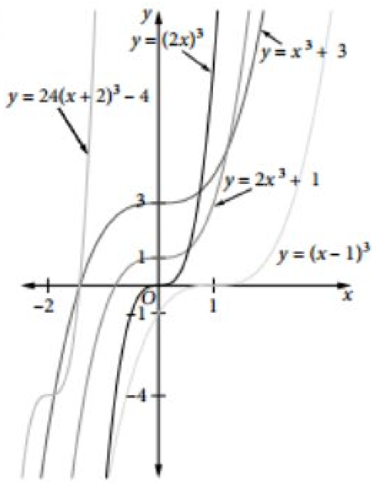
(e) $3f(2(x+2)) - 4 = 3[2(x+2)]^3 - 4 = 24(x+2)^3 - 4$

3 (a) $f(2x) = \cos x$ (b) $f\left(x + \frac{\pi}{3}\right) = \cos \frac{1}{2}\left(x + \frac{\pi}{3}\right)$

(c) $2f(x) = 2\cos \frac{x}{2}$ (d) $f(x) - 1 = \cos \frac{x}{2} - 1$

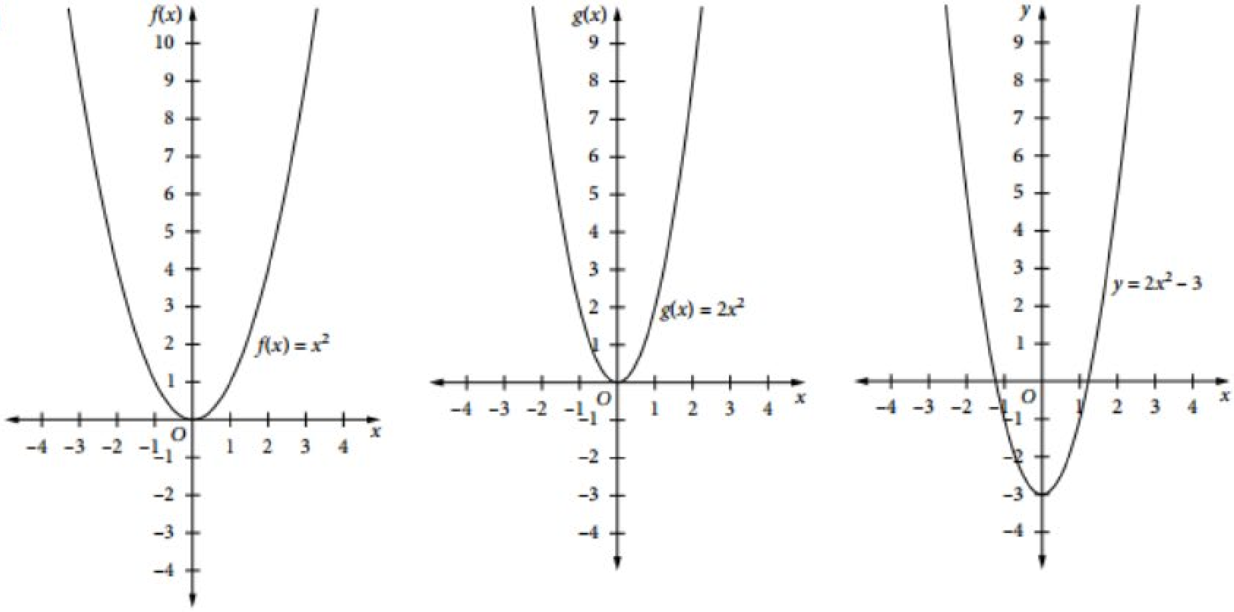
(e) $2f\left(x + \frac{\pi}{6}\right) + 1 = 2\cos \frac{1}{2}\left(x + \frac{\pi}{6}\right) + 1$

4 (a)–(e)

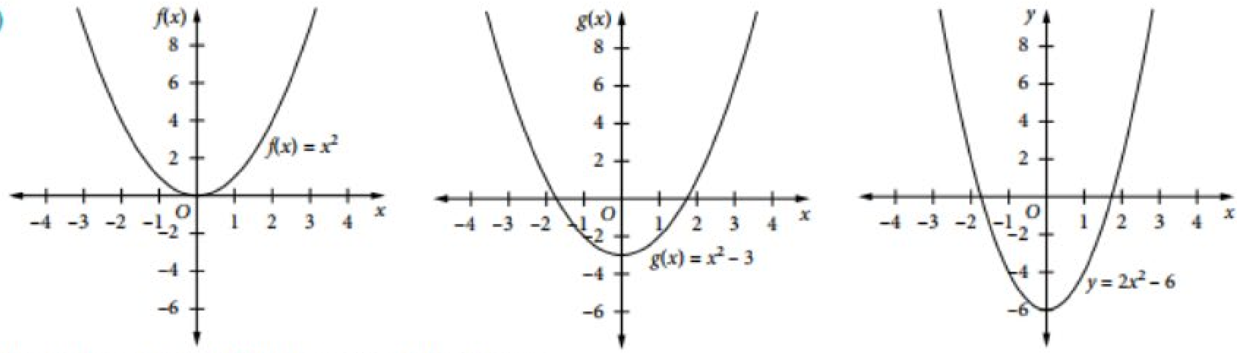


5 A

6 (a)

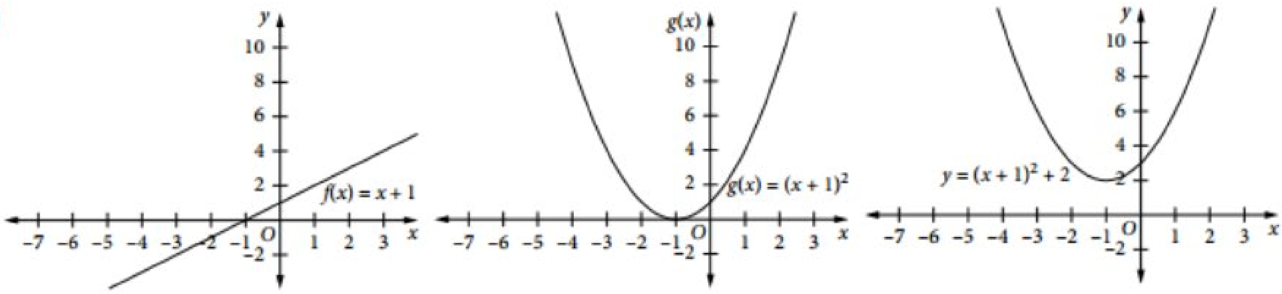


(b)

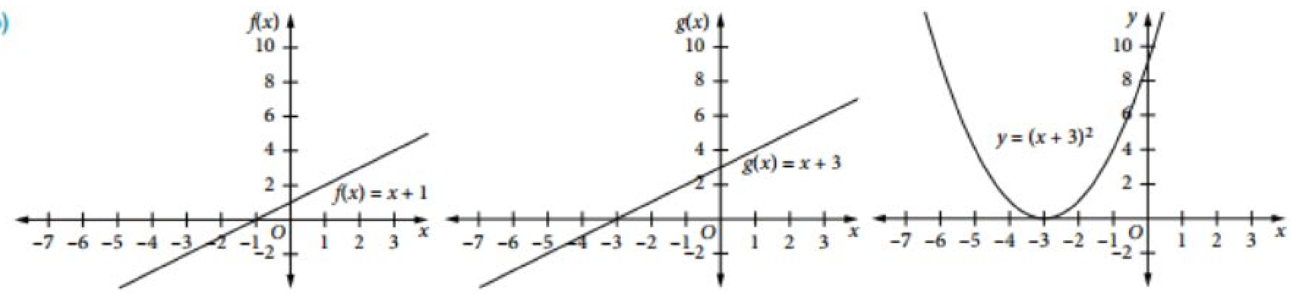


(c) The final graph in part (b) has been moved down 6 units, whereas part (a) was moved down 3 units. They have the same dilation.

7 (a)

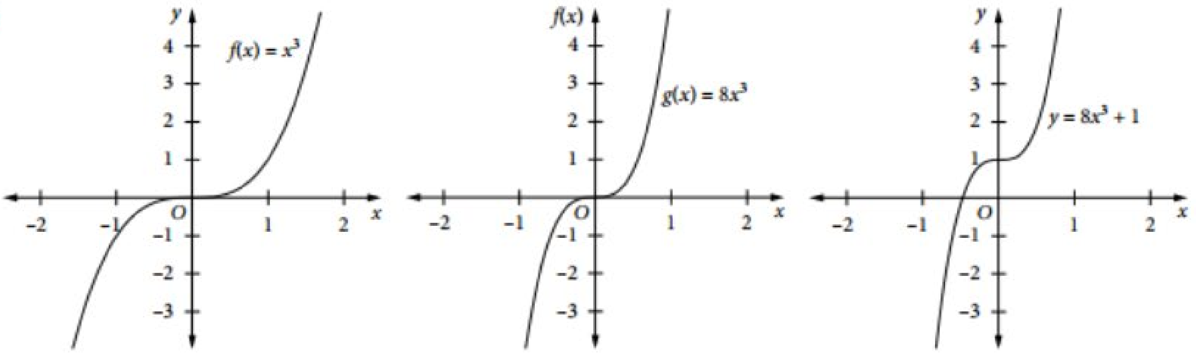


(b)

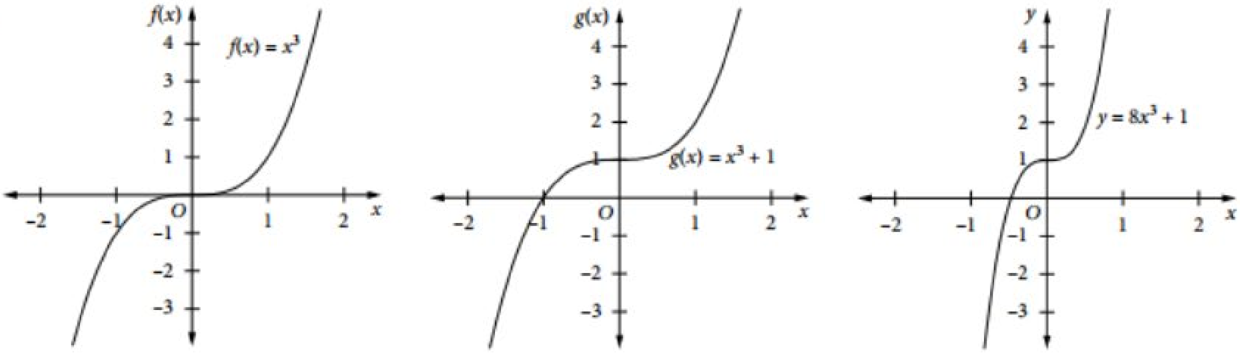


(c) Both curves are parabolas. The vertex in part (a) is at $(-1, 2)$, whereas the vertex in part (b) is at $(-3, 0)$.

8 (a)

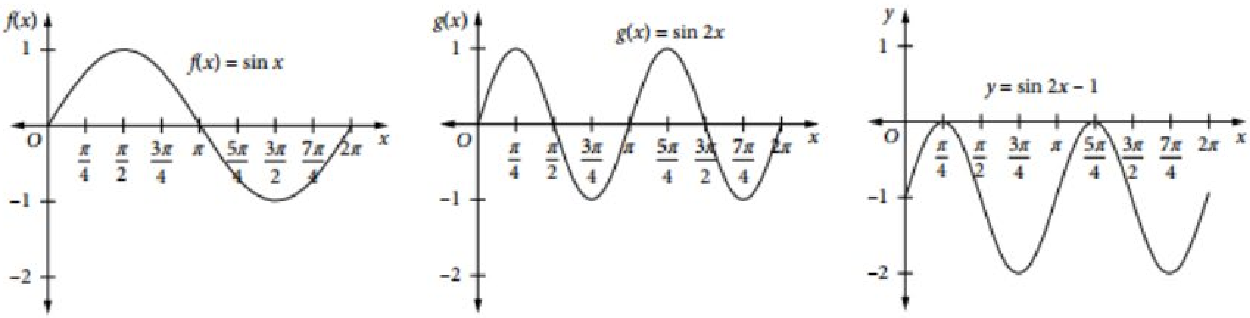


(b)

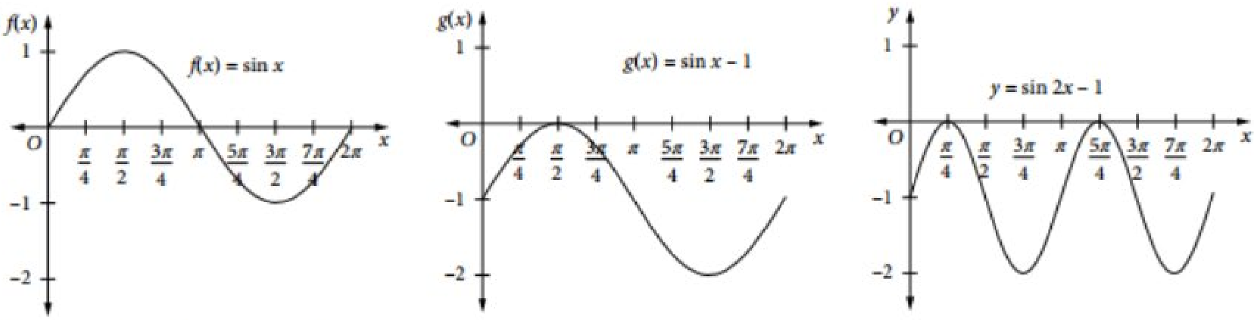


(c) The final graph is the same in each case.

9 (a)

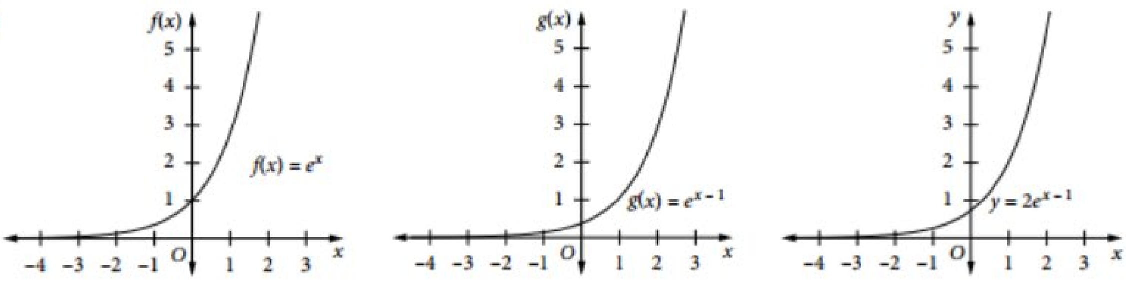


(b)

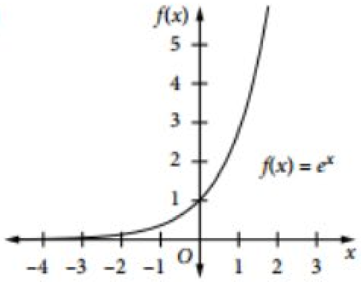


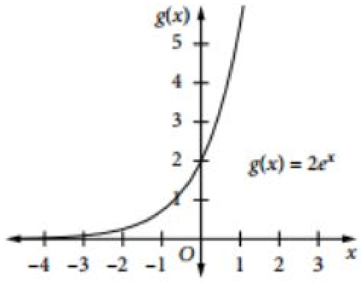
(c) The final graph is the same in each case.

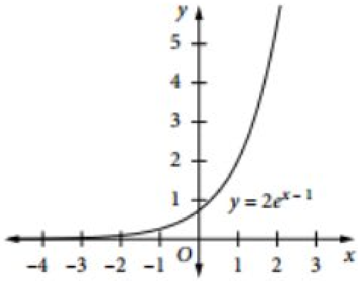
10 (a)



(b)





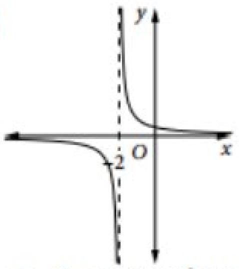


(c) The final graph is the same in each case.

EXERCISE 15.4

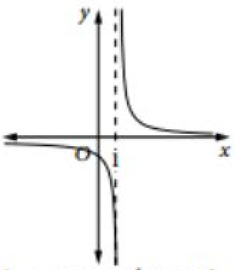
1 A

2 (a)



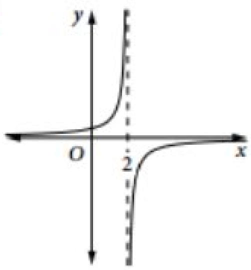
$x < -2$ range: real y, $y \neq 0$

(b)



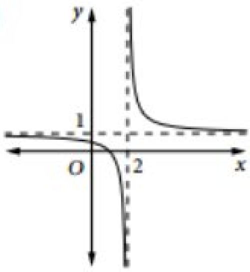
$x < 1$ range: real y, $y \neq 0$

(c)

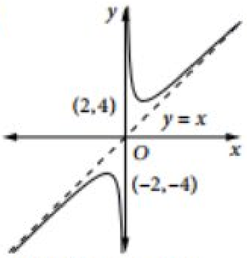


$x > 2$ range: real y, $y \neq 0$

3 (a) $1 + \frac{1}{x-2} = \frac{x-2+1}{x-2} = \frac{x-1}{x-2}$ (b)

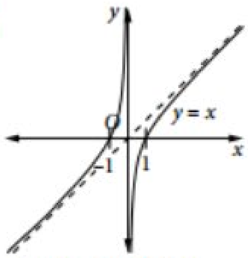


4 (a)



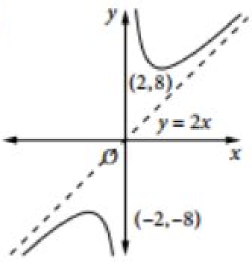
no inflections, $x > 0$
range: real y, $|y| \geq 2$

(b)



no turning points,
no inflections, $x < 0$
range: real y

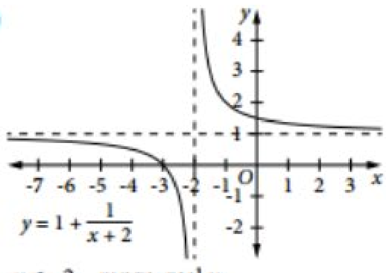
(c)



no inflections, $x > 0$
range: real y, $|y| \geq 8$

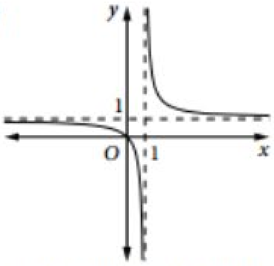
5 (a) correct (b) incorrect (c) correct (d) correct

6 (a)



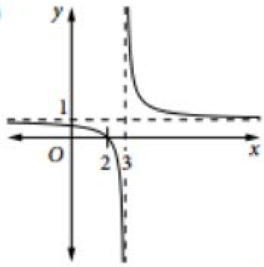
$x < -2$ range: real y

(b)



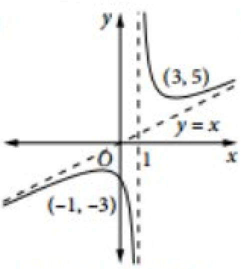
$x < 1$ range: real y, $y \neq 1$

(c)



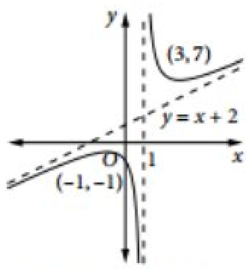
$x < 3$ range: real y, $y \neq 1$

7 (a)



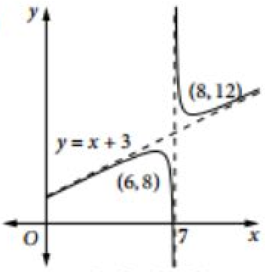
range: $y \leq -3$, $y \geq 5$

(b)



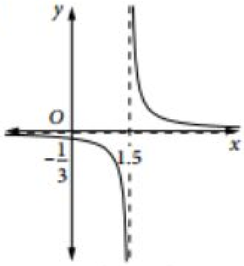
range: $y \leq -1$, $y \geq 7$

(c)



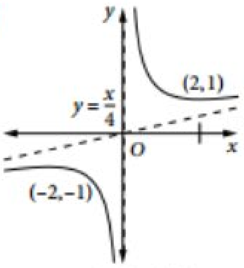
range: $y \le -8, y \ge 12$

(d)



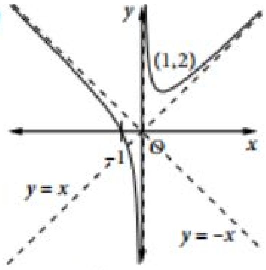
range: real $y, y \ne 0$

(e)



range: $y \le -1, y \ge 1$

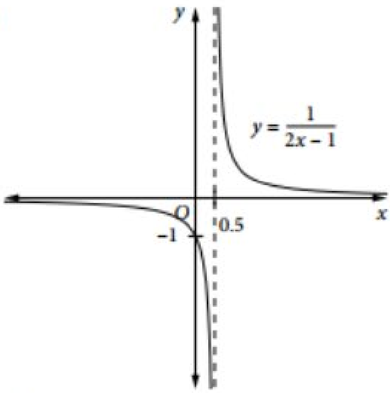
(f)



range: real y

EXERCISE 15.5

1 (a)



(b) $\dfrac{dy}{dx} = \dfrac{-2}{(2x-1)^2}$

$x = 1$: $\dfrac{dy}{dx} = -2$

$x = 1, y = 1$

Equation of tangent: $y - 1 = -2(x - 1)$

$2x + y - 3 = 0$

(c) $x = -1$: $\dfrac{dy}{dx} = -\dfrac{2}{9}$

Gradient of normal $= \dfrac{9}{2}$

$x = -1, y = -\dfrac{1}{3}$

Equation of normal: $y + \dfrac{1}{3} = \dfrac{9}{2}(x + 1)$

$6y + 2 = 27x + 27$

$27x - 6y + 25 = 0$

(d) $2x + y - 3 = 0$ [1]

$27x - 6y + 25 = 0$ [2]

[1] × 6: $12x + 6y - 18 = 0$ [3]

[2] + [3]: $39x + 7 = 0$

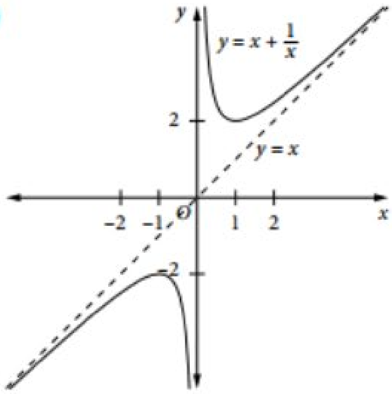
$x = -\dfrac{7}{39}$

Substitute into [1]: $-\dfrac{14}{39} + y - 3 = 0$

$y = \dfrac{131}{39}$

Point of intersection is $\left(-\dfrac{7}{39}, \dfrac{131}{39}\right)$

2 (a)



(b) $\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2} = \dfrac{x^2-1}{x^2} = \dfrac{(x+1)(x-1)}{x^2}$

For stationary points, $\dfrac{dy}{dx} = 0$: $x = \pm 1$

$x = 1, y = 2$. $x = -1, y = -2$.

$\dfrac{d^2y}{dx^2} = \dfrac{2}{x^3}$

$x = 1$: $\dfrac{d^2y}{dx^2} = 2 > 0$

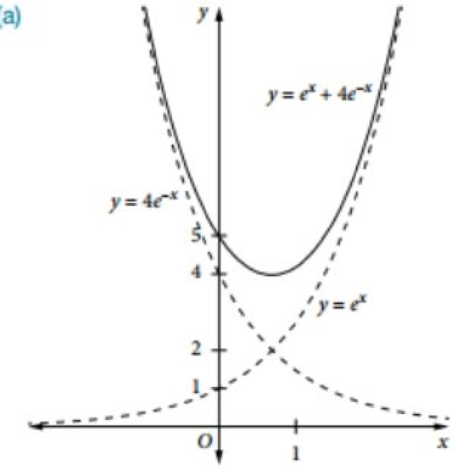
Minimum turning point at (1, 2)

$x = -1$: $\dfrac{d^2y}{dx^2} = -2 < 0$

Maximum turning point at (−1,−2)

(c) 2

3 (a)



On the left the curve $y = 4e^{-x}$ is the asymptote, on the right the curve $y = e^x$ is the asymptote.

(b) $f'(x) = e^x - 4e^{-x} = \dfrac{e^{2x}-4}{e^x}$

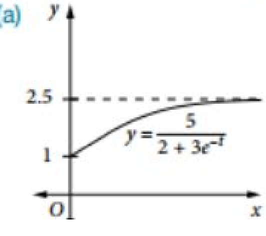
For stationary points, $\dfrac{dy}{dx} = 0$: $e^{2x} = 4$, $e^x = \pm 2$. Since $e^x > 0$, $e^x = 2$ is the only solution.

$e^x = 2 \to x = \ln 2$

$f'(x) > 0$ for $x > \ln 2$

(c) The minimum value of $f(x)$ is 4 and it occurs when $x = \ln 2$.

4 (a)



(b) $f'(t) = \dfrac{15e^{-t}}{\left(2+3e^{-t}\right)^2}$

$e^{-t} > 0$ for all values of t, the denominator is always positive so $f'(t) > 0$ for $t \geq 0$.

(c) As $t \to \infty$ then $f(t) \to 2.5$.

(d) $1 \leq f(t) < 2.5$

5 (a) $f(0) = 1, f\left(\dfrac{\pi}{2}\right) = 0, f(\pi) = -e^{-\pi}$

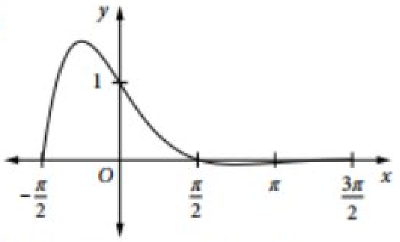
(b) $f'(x) = -e^{-x}\cos x - e^{-x}\sin x = -e^{-x}(\cos x + \sin x)$

(c) $f'(0) = -1 \times 1 - 1 \times 0 = -1$

$$f'\left(\frac{3\pi}{4}\right) = -e^{-\frac{3\pi}{4}\left(\cos\frac{3\pi}{4}+\sin\frac{3\pi}{4}\right)} = -e^{-\frac{3\pi}{4}\left(\frac{-1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\right)} = 0$$

$$f'\left(-\frac{\pi}{4}\right) = -e^{\frac{\pi}{4}\left(\cos-\frac{\pi}{4}+\sin-\frac{\pi}{4}\right)} = -e^{-\frac{\pi}{4}\left(\frac{1}{\sqrt{2}}+\frac{-1}{\sqrt{2}}\right)} = 0$$

(d)



(e) $f''(x) = e^{-x}(\cos x + \sin x) - e^{-x}(-\sin x + \cos x) = 2e^{-x}\sin x$

$x = -\dfrac{\pi}{4}, f''(x) = 2e^{\frac{\pi}{4}\sin\left(-\frac{\pi}{4}\right)} = -\sqrt{2}e^{\frac{\pi}{4}<0}$

Maximum turning point when $x = -\dfrac{\pi}{4}$.

$f\left(-\dfrac{\pi}{4}\right) = e^{\frac{\pi}{4}\cos\left(-\frac{\pi}{4}\right)} = \dfrac{e^{\frac{\pi}{4}}}{\sqrt{2}} \approx 1.55$

6 (a) $f(x) = \log_e(\sin x)$. Require $\sin x > 0$, so $0 < x < \pi$.

(b) $f'(x) = \dfrac{\cos x}{\sin x} = \cot x$. Domain is $0 < x < \pi$.

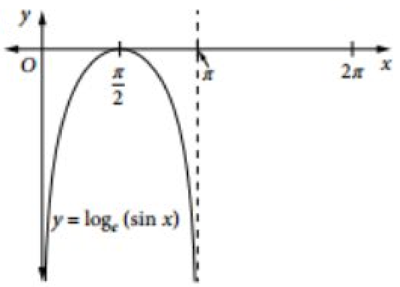
(c) $f'(x) = 0$ when $\cot x = 0$ so $x = \dfrac{\pi}{2}$.

$f''(x) = -\text{cosec}^2\, x$

$f''\left(\dfrac{\pi}{2}\right) = -\text{cosec}^2\,\dfrac{\pi}{2} = -1 < 0$

Maximum value of $f(x)$ when $x = \dfrac{\pi}{2}$ is $\log_e\left(\sin\dfrac{\pi}{2}\right) = 0$

(d)



7 (a) $y = \log_e(1 + \sin x)$

$$\frac{dy}{dx} = \frac{\cos x}{1+\sin x}$$

$$\frac{d^2y}{dx^2} = \frac{-\sin x(1+\sin x) - \cos x \times \cos x}{(1+\sin x)^2}$$

$$= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2} = \frac{-(1+\sin x)}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

(b) $\dfrac{dy}{dx} = 0$ when $\cos x = 0$, $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}, \dfrac{5\pi}{2}, \dfrac{7\pi}{2}$ but y is undefined for $x = \dfrac{3\pi}{2}, \dfrac{7\pi}{2}$.

Hence $x = \dfrac{\pi}{2}, \dfrac{5\pi}{2}$

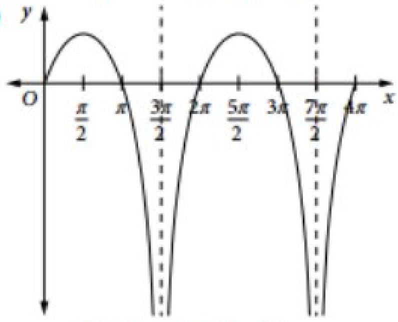
(c) $x = \dfrac{\pi}{2}$: $\dfrac{d^2y}{dx^2} = \dfrac{-1}{1+1} = -\dfrac{1}{2} < 0$ so maximum when $x = \dfrac{\pi}{2}$.

$x = \dfrac{5\pi}{2}$: $\dfrac{d^2y}{dx^2} = \dfrac{-1}{1+1} = -\dfrac{1}{2} < 0$ so maximum when $x = \dfrac{5\pi}{2}$.

(d) $\dfrac{-1}{1+\sin x} \neq 0$ for any value of x so $\dfrac{d^2y}{dx^2} \neq 0$ for any value of x.

Hence no points of inflection.

(e)



8 $C(t) = 1000\left[\cos\left(\dfrac{\pi(t-8)}{2}\right) + 2\right]^2 - 1000$, for $8 \leq t \leq 16$

(a) $\dfrac{dC}{dt} = 1000 \times 2\left[\cos\left(\dfrac{\pi(t-8)}{2}\right) + 2\right] \times \left[-\sin\left(\dfrac{\pi(t-8)}{2}\right)\right] \times \dfrac{\pi}{2}$

$= -1000\pi\sin\left(\dfrac{\pi(t-8)}{2}\right)\left[\cos\left(\dfrac{\pi(t-8)}{2}\right) + 2\right]$

$\dfrac{dC}{dt} = 0$: $\sin\left(\dfrac{\pi(t-8)}{2}\right) = 0$ or $\cos\left(\dfrac{\pi(t-8)}{2}\right) = -2$

$\dfrac{\pi(t-8)}{2} = 0, \pi, 2\pi, 3\pi, 4\pi$

$t - 8 = 0, t - 8 = 2, t - 8 = 4, t - 8 = 6, t - 8 = 8$

$t = 8, 10, 12, 14, 16$

$t = 8, 12, 16$: $\cos\left(\dfrac{\pi(t-8)}{2}\right) = 1$

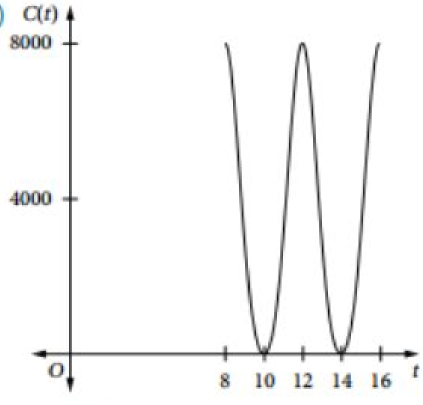
$t = 10$: $\cos\left(\dfrac{\pi(t-8)}{2}\right) = \cos\pi = -1$

$t = 14$: $\cos\left(\dfrac{\pi(t-8)}{2}\right) = \cos 3\pi = -1$

The least value of $C(t)$ will occur when $t = 10, 14$.

Minimum concentration $= 1000[\cos\pi + 2]^2 - 1000 = 0$

(b)



9 (a) $h^2 + \left(\dfrac{x}{2}\right)^2 = 1.69x^2$

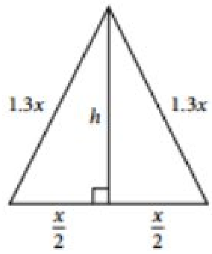
$h^2 = 1.69x^2 - 0.25x^2$

$h^2 = 1.44x^2$

$h = 1.2x$

(b) $A = xy + \dfrac{1}{2} \times x \times 1.2x$

$= xy + 0.6x^2$

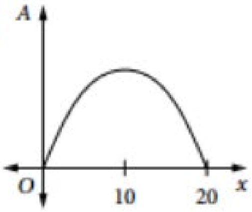


(c) $2y + x + 2.6x = 48$

$2y + 3.6x = 48$

$y = 24 - 1.8x$

(d) $A(x) = x \times (24 - 1.8x) + 0.6x^2$
$= 24x - 1.2x^2$

(e) 

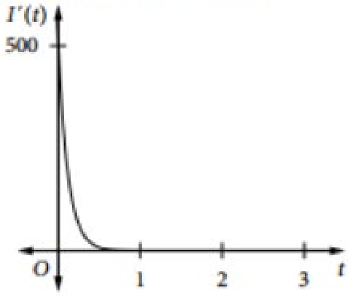
(f) Maximum area when $x = 10$ m, $y = 6$ m, equal sides of the isosceles triangle 13 m.
Maximum area $= 120\ \text{m}^2$

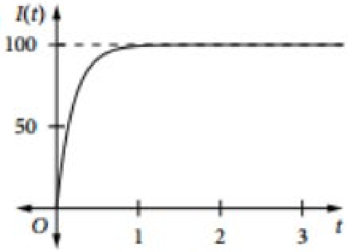
10 $I(t) = 100(1 - e^{-5t})$

(a) $I(0) = 100(1 - 1) = 0$
$I(0.2) = 100(1 - e^{-1}) = 63.2$ amps
$I(1) = 100(1 - e^{-5}) = 99.3$ amps

(b) As t increases the current approaches 100 amps.

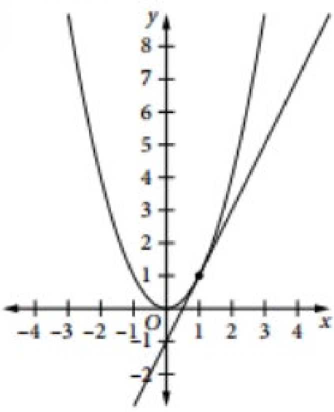
(c) $I'(t) = 100 \times 5e^{-5t} = 500e^{-5t}$



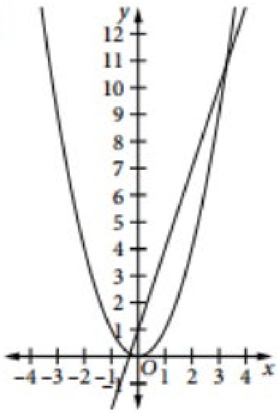
(d) 

(e) The graph of $I'(t)$ shows the gradient of $I(t)$ over time. The gradient graph shows the current increasing rapidly at the start then staying practically the same.

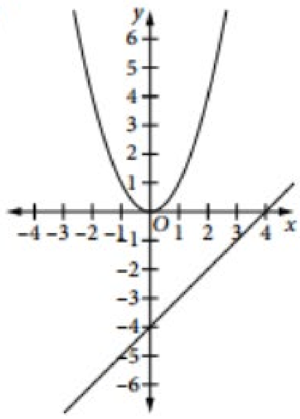
EXERCISE 15.6

1 (a) 

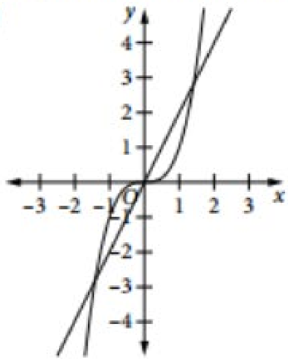
Graphs touch, equation has one solution.

(b) 

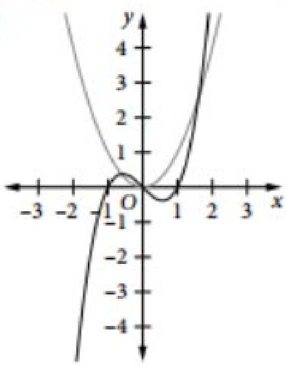
Graphs intersect twice, equation has two solutions.

(c) 

Graphs do not intersect so equation has no solutions

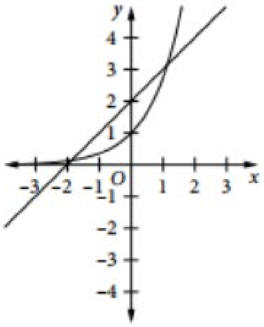
(d) 

Graphs intersect 3 times. Equation has 3 solutions.

(e) 

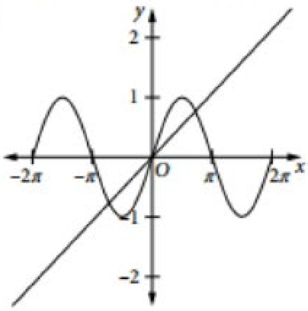
Graphs intersect 3 times. Equation has 3 solutions.

(f)



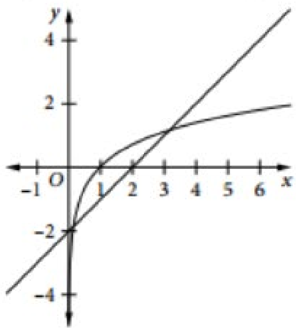
Graphs intersect twice, equation has two solutions.

2 (a)



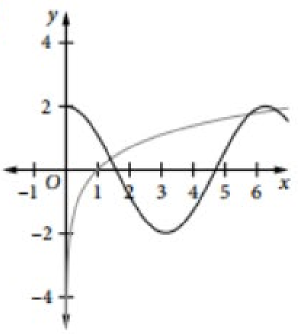
Graphs intersect 3 times. Equation has 3 solutions.

(b)



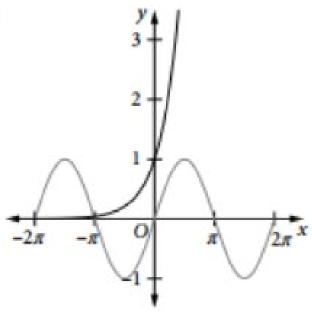
Graphs intersect twice, equation has two solutions.

(c)



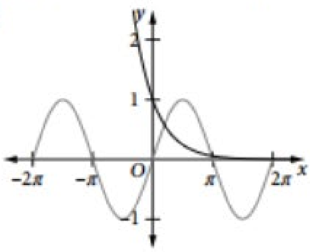
Graphs intersect twice, equation has two solutions.
If the domain of 2cos x had been $0 \le x \le 3\pi$, the equation would have had 3 solutions.

(d)



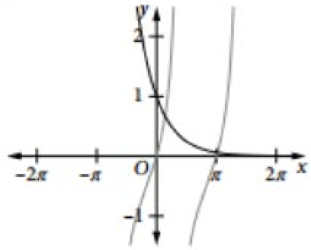
Graphs intersect 3 times, equation has 3 solutions.

(e)



Graphs intersect twice, equation has two solutions.

(f)

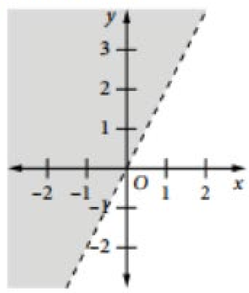


Graphs intersect twice, equation has two solutions.

3 2.84

EXERCISE 15.7

1



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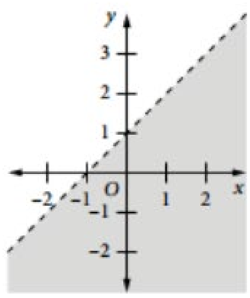

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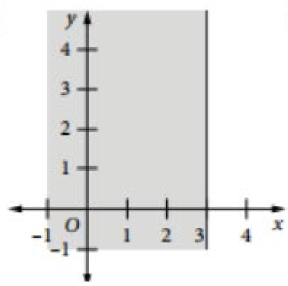

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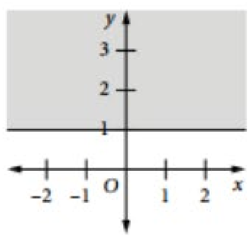

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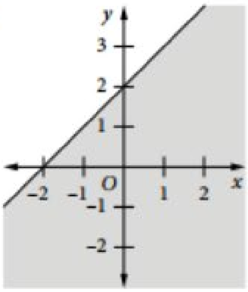

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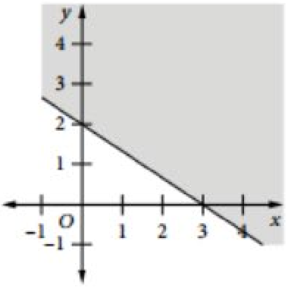

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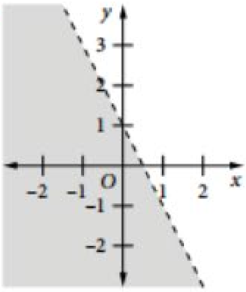

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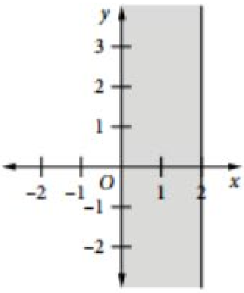

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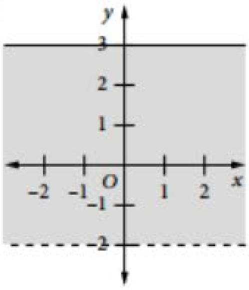

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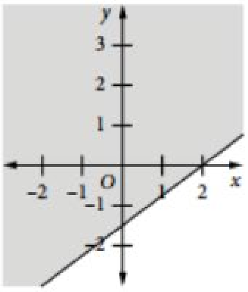

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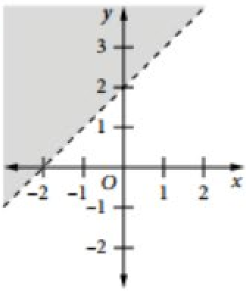

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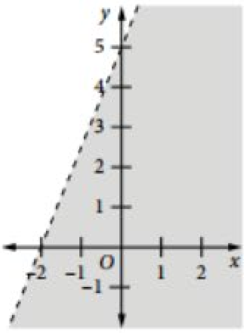

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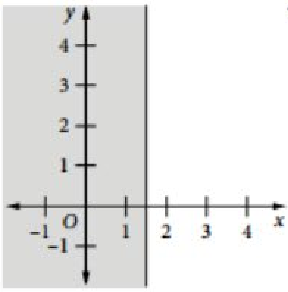

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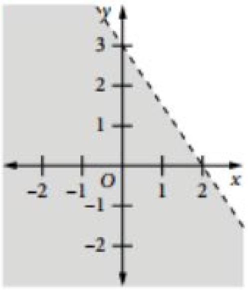

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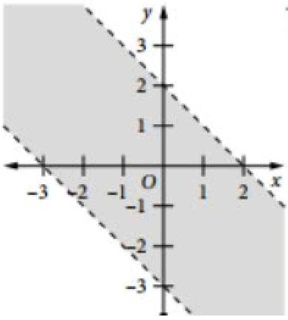

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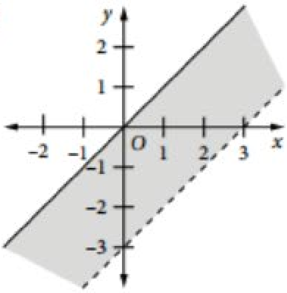
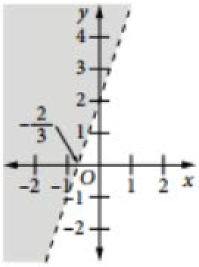
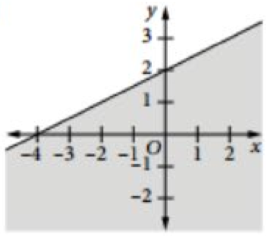
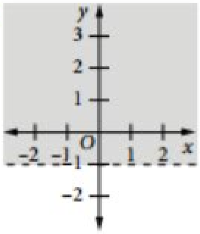
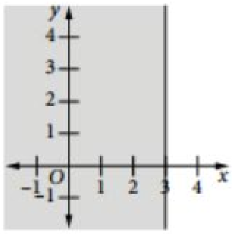


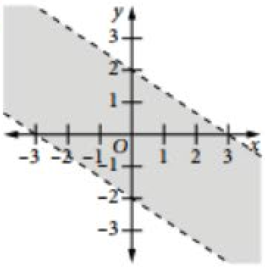
17 A

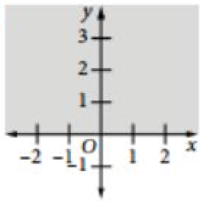
18 (a)


(b)


(c)


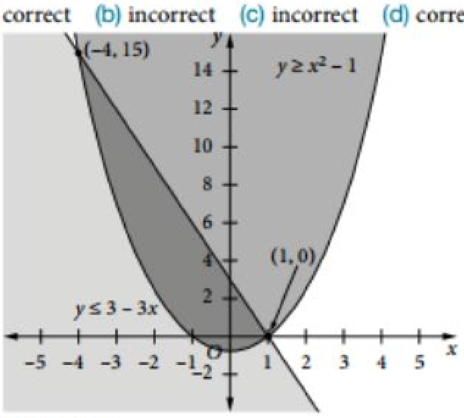
(d)


(e)


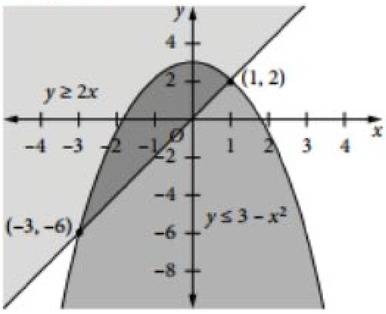
(f)


19 (a) The region below the line $y = x + 2$.
(b) The region on and above the line $y = x$.
(c) The region to the right of the line $x = 3$.
(d) The region on and below the line $y = 4$.
(e) The region on and below the line $x + 3y = 9$.
(f) The region on and to the right of the line $x = -2$ that is also to the left of the line $x = 3$.

20 (a) correct (b) incorrect (c) incorrect (d) correct

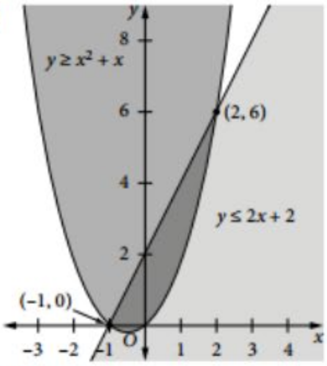
21 (a)


(b) $-4 \leq x \leq 1$

22 (a)


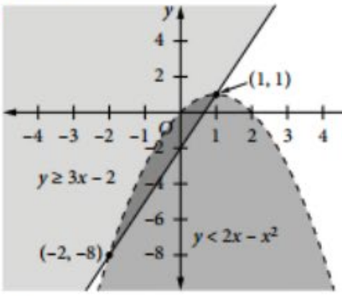
(b) $-3 \leq x \leq 1$

23 (a)



(b) $-1 \leq x \leq 2$

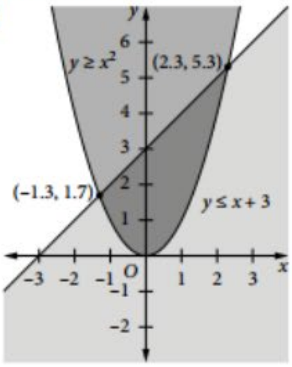
24 (a)



(b) $x^2 + x - 2 < 0$
$3x - 2 < 2x - x^2$
$-2 < x < 1$

(c) $x^2 + x - 2 < 0$
$(x + 2)(x - 1) < 0$
Roots: $x = -2, 1$
test using $x = 0$
$0 + 0 - 2 < 0$
$-2 < 0$: true
Solution is $-2 < x < 1$

25 (a)



(b) $x^2 \leq x + 3$
$x^2 - x - 3 \leq 0$
$-1.3 \leq x \leq 2.3$

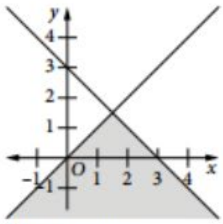
(c) Solve $x^2 - x - 3 = 0$
$x = \dfrac{1 \pm \sqrt{13}}{2}$
test using $x = 0$
$0 - 0 - 3 < 0$
$-3 < 0$: true
Solution is $\dfrac{1 - \sqrt{13}}{2} \leq x \leq \dfrac{1 + \sqrt{13}}{2}$
$-1.303 \leq x \leq 2.303$

EXERCISE 15.8

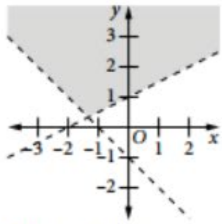
1 (a) The region on and to the right of the line $x = -1$ that is also below the line $y = 2$.
(b) The region on and above the line $x + y = 1$ that is also on and below the line $y = x + 1$.
(c) The region above the line $y = 1$ that is also above the line $x + y = 1$.
(d) The region on and to the right of the line $x = -1$ that is also to the left of the line $x = 2$.
(e) The region on and below the line $y = 2x + 2$ that is also below the line $x + y = 2$.
(f) The region on and above the line $y = x$ that is also on and below the line $y = 2x$.

2 C 3 $x + y \geq 3, y \leq x + 1$

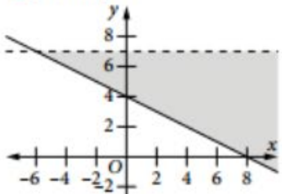
4 (a) yes, no, yes



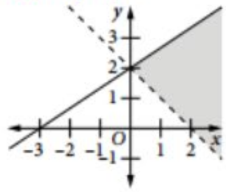
(b) no, no, yes



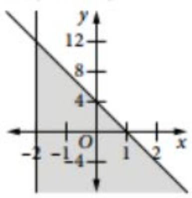
(c) yes, no, yes



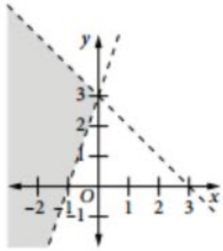
(d) no, yes, yes



(e) yes, no, yes

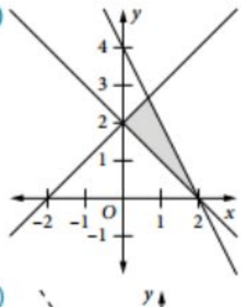


(f) no, no, no



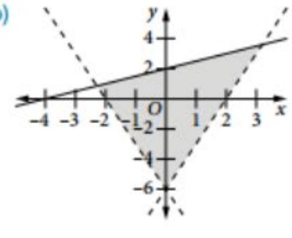
5 (a) correct (b) incorrect (c) correct (d) correct

6 (a)



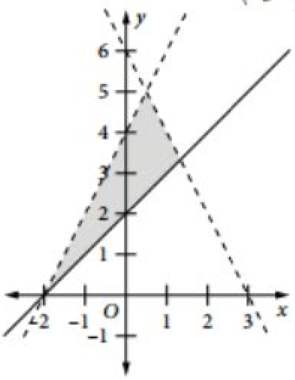
$y = x + 2, 2x + y = 4;$
$2x + x + 2 = 4, 3x = 2:$
$x = \frac{2}{3}, y = 2\frac{2}{3}$
vertices: $\left(\frac{2}{3}, 2\frac{2}{3}\right), (2,0), (0,2)$

(b)

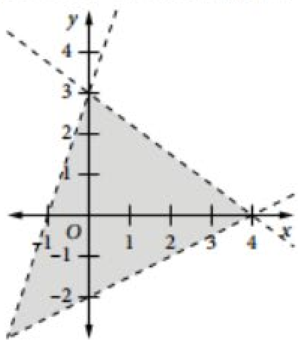


$2y - x = 4, y = 3x - 6;$
$6x - 12 - x = 4, 5x = 16:$
$x = 3\frac{1}{5}, y = 3\frac{3}{5}$
$2y - x = 4, 3x + y = -6;$
$6x - 12 - x = 4, 7x = -16:$
$x = -2\frac{2}{7}, y = \frac{6}{7}$
vertices: $\left(3\frac{1}{5}, 3\frac{3}{5}\right), \left(-2\frac{2}{7}, \frac{6}{7}\right), (0, -6)$

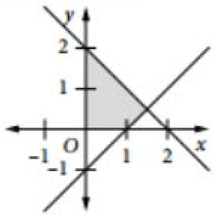
(c) vertices: $(-2, 0)$, $(0.5, 5)$, $\left(1\frac{1}{3}, 3\frac{1}{3}\right)$



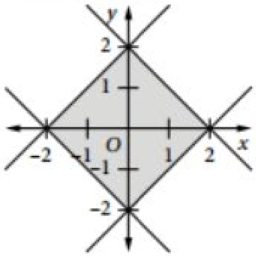
(d) vertices: $(-2, -3)$, $(4, 0)$, $(0, 3)$



(e) vertices: $(0, 0)$, $(1, 0)$, $(1.5, 0.5)$, $(0, 2)$



(f) vertices: $(-2, 0)$, $(0, 2)$, $(2, 0)$, $(0, -2)$



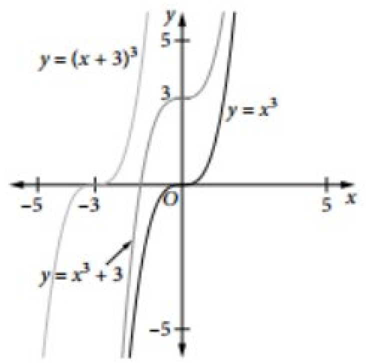
7 (a) the region bounded by the lines $y = x$, $x = 2$ and $y = 1$; $y \le x$, $x \le 2$, $y \ge 1$; $A(2, 2)$, $B(2, 1)$, $C(1, 1)$

(b) the region bounded by the lines $y = 2x$, $y = x$ and $x + y = 3$; $y \le 2x$, $y \ge x$, $x + y \le 3$; $A(0, 0)$, $B(1, 2)$, $C(1.5, 1.5)$

(c) the region bounded by the lines $y = \frac{x}{2} + 1$, $y = \frac{x}{6} + 1$ and $x + 2y = 6$; $y \le \frac{x}{2} + 1$, $y \ge \frac{x}{6} + 1$, $x + 2y \le 6$; $A(0, 1)$, $B(2, 2)$, $C(3, 1.5)$

CHAPTER REVIEW 15

1



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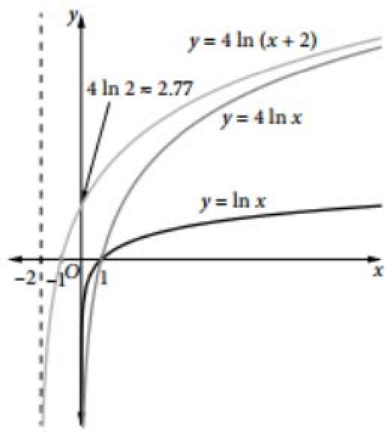
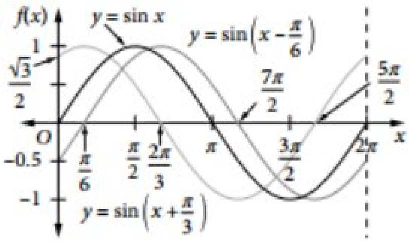
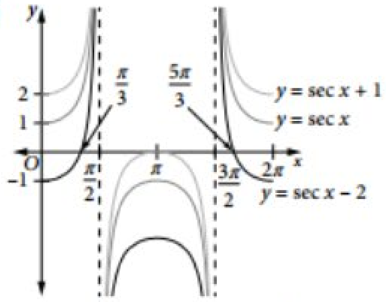


The dilation from the x-axis for the second and third graphs has factor 4.

3 (a)



(b)

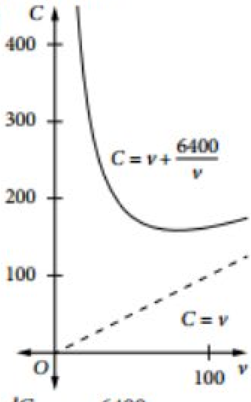


4 (a) $f(2x) = e^{2x}$ (b) $f(x - 3) = e^{x-3}$ (c) $f(x) + 1 = e^x + 1$
(d) $2f(x) + 4 = 2e^x + 4$ (e) $f(2(x + 2)) - 1 = e^{2(x+2)} - 1$

5 $C = v + \frac{6400}{v}$

(a) $0 < v \le 110$

(b)



(c) $\frac{dC}{dt} = 1 - \frac{6400}{v^2}$

$\frac{dC}{dt} = 0 : 1 - \frac{6400}{v^2} = 0$

$v^2 = 6400$

$v = 80$

Average speed is $80\ \text{km h}^{-1}$.

6 (a) Let $BC = y$ cm

$AB = AE = EB = DC = x$ cm

$BC = BF = FC = AD = y$ cm

$3x + 3y = 54$

$x + y = 18$

$y = 18 - x$

$$A = xy + \frac{1}{2}x^2 \sin 60° + \frac{1}{2}y^2 \sin 60°$$

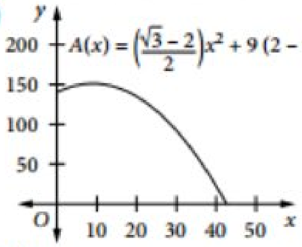
$$A(x) = x(18 - x) + x^2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2} + (18 - x)^2 \times \frac{1}{2} \times \frac{\sqrt{3}}{2}$$

$$= 18x - x^2 + \frac{\sqrt{3}}{4}x^2 + 81\sqrt{3} - 9\sqrt{3}x + \frac{\sqrt{3}}{4}x^2$$

$$= \frac{\left(\sqrt{3} - 2\right)}{2}x^2 + 9\left(2 - \sqrt{3}\right)x + 81\sqrt{3}$$

(b) The domain is $0 < x < 18$.

(c)



(d) $\frac{dA}{dx} = \left(\sqrt{3} - 2\right)x + 9\left(2 - \sqrt{3}\right)$

$\frac{dA}{dx} = 0$: $\left(\sqrt{3} - 2\right)x + 9\left(2 - \sqrt{3}\right) = 0$

$x = 9$

The rectangle becomes a square of side 9 cm when the area is a maximum.

7 (a) (i) $f(x) = 4 + \frac{3x - 1}{x^2}$

$4 + \frac{3x - 1}{x^2} = 0$

$4x^2 + 3x - 1 = 0$

$(4x - 1)(x + 1) = 0$

$x = -1, \frac{1}{4}$

(ii) As $x \to \infty, f(x) \to 4$

(iii) $f(x)$ is undefined at $x = 0$, it approaches $-\infty$.

(iv) Asymptotes are $x = 0$ and $y = 4$

(v) $f'(x) = 0 + \frac{3x^2 - (3x - 1) \times 2x}{x^4} = \frac{3x - 6x + 2}{x^3} = \frac{2 - 3x}{x^3}$

$f'(x) = 0$: $x = \frac{2}{3}, y = 6\frac{1}{4}$

$f''(x) = \frac{-3x^3 - (2 - 3x) \times 3x^2}{x^6} = \frac{-3x - 6 + 9x}{x^4} = \frac{6(x - 1)}{x^4}$

$f''\left(\frac{2}{3}\right) = 6 \times \frac{3^4}{2^4} \times \left(-\frac{1}{3}\right) < 0$

$\left(\frac{2}{3}, 6\frac{1}{4}\right)$ is a maximum turning point.

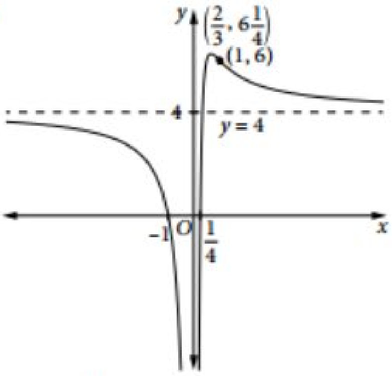
(vi) $f''(x) = 0$ when $x = 1$.

$f''\left(\frac{2}{3}\right) < 0$

$f''(2) = 6 \times \frac{1}{16} > 0$

Concavity changes at $x = 1$ so $(1, 6)$ is a point of inflection.

(b)



8 (a) If $y = \frac{4x}{(x - 1)^2}$

(i) $\frac{dy}{dx} = \frac{4(x - 1)^2 - 4x \times 2(x - 1)}{(x - 1)^4} = \frac{-4(x + 1)}{(x - 1)^3}$

For stationary points, $\frac{dy}{dx} = 0$: $x = -1, y = -1$

$$\frac{d^2y}{dx^2} = \frac{-4\left((x - 1)^3 - (x + 1) \times 3(x - 1)^2\right)}{(x - 1)^6}$$

$$= \frac{-4(x - 1 - 3x - 3)}{(x - 1)^4} = \frac{8(x + 2)}{(x - 1)^4}$$

$x = -1$: $\frac{d^2y}{dx^2} = \frac{8}{16} > 0$

Minimum turning point at $(-1, -1)$.

(ii) $\frac{d^2y}{dx^2} = 0$: $x = -2, y = -\frac{8}{9}$.

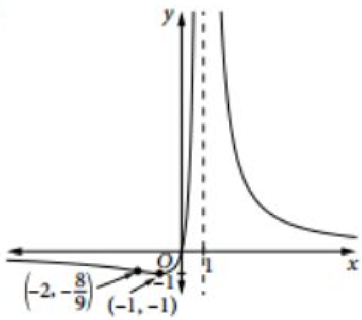
$x = -1$: $\frac{d^2y}{dx^2} > 0$

$x = -3$: $\frac{d^2y}{dx^2} = \frac{-8}{4^4} < 0$

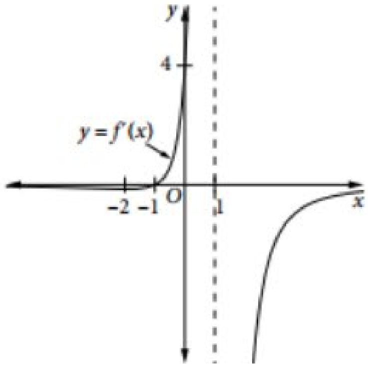
Concavity changes at $x = -2$ so $\left(-2, \frac{-8}{9}\right)$ is a point of inflection.

(iii) $x = 1, y = 0$

(b)



(c) $f'(x) = \frac{-4(x + 1)}{(x - 1)^3}$



9 (a) $\frac{x(x+1)}{x-1} = \frac{x^2+x}{x-1}$, $px + q + \frac{r}{x-1} = \frac{px^2+(q-p)x+(r-q)}{x-1}$,

$p = 1, q = 2, r = 2$;

$\frac{x(x+1)}{x-1} = x + 2 + \frac{2}{x-1}$

(b) $y = \frac{x(x+1)}{x-1} = x + 2 + \frac{2}{x-1}$,

$\frac{dy}{dx} = 1 - \frac{2}{(x-1)^2}$

$\frac{dy}{dx} = 0$: $(x-1)^2 = 2, x - 1 = \pm\sqrt{2}, x = 1 \pm \sqrt{2}$

$x = 1 + \sqrt{2}, y = 1 + \sqrt{2} + 2 + \frac{2}{\sqrt{2}} = 3 + 2\sqrt{2}$

$x = 1 - \sqrt{2}, y = 1 - \sqrt{2} + 2 + \frac{2}{-\sqrt{2}} = 3 - 2\sqrt{2}$

$\frac{d^2y}{dx^2} = \frac{4}{(x-1)}$

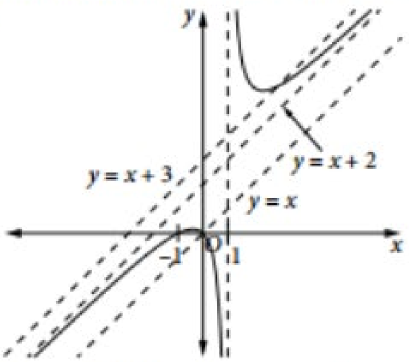
$x = 1 + \sqrt{2}$: $\frac{d^2y}{dx^2} = \frac{4}{2\sqrt{2}} > 0$

Minimum turning point at $(1 + \sqrt{2}, 3 + 2\sqrt{2})$

$x = 1 - \sqrt{2}$: $\frac{d^2y}{dx^2} = \frac{4}{-2\sqrt{2}} < 0$

Maximum turning point at $(1 - \sqrt{2}, 3 - 2\sqrt{2})$

(c), (d)



(e) $c = 2$. $y = x + 2$ is the sloping asymptote.

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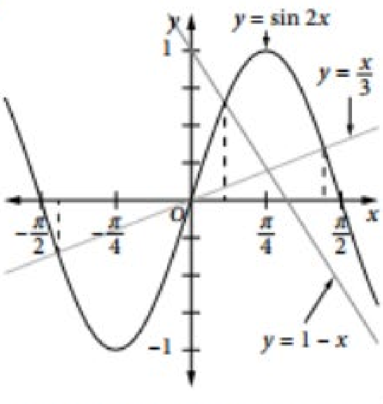
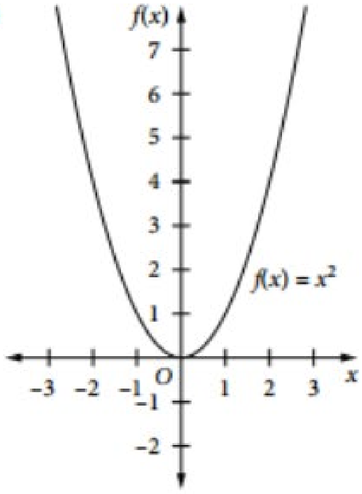
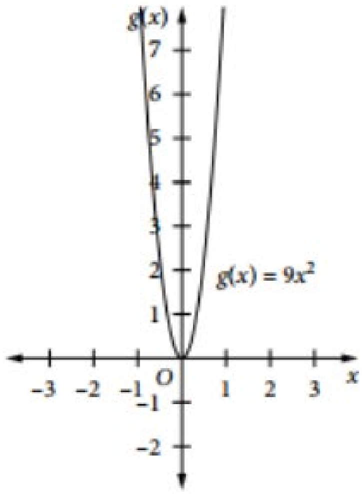


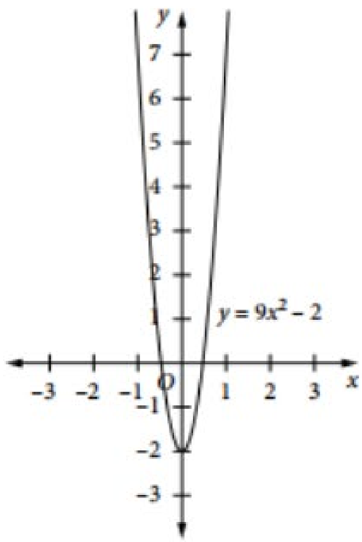
(a) The graphs $y = \sin 2x$ and $y = \frac{x}{3}$ intersect at three places: when $x = 0$ and near $x = \pm 0.4\pi$ or ± 1.3.
More accurate solutions using technology give $x = 0, \pm 0.43\pi$ or ± 1.34.

(b) The graphs $y = \sin 2x$ and $y = 1 - x$ intersect at one place: near $x = 0.1\pi$ or 0.3.
A more accurate solution using technology gives $x = 0.11\pi$ or 0.35.

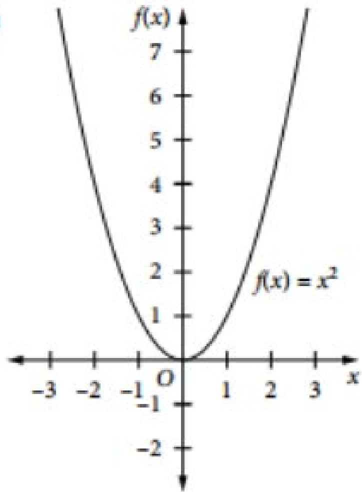
11 (a)

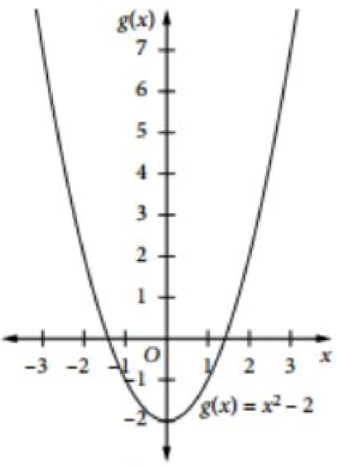


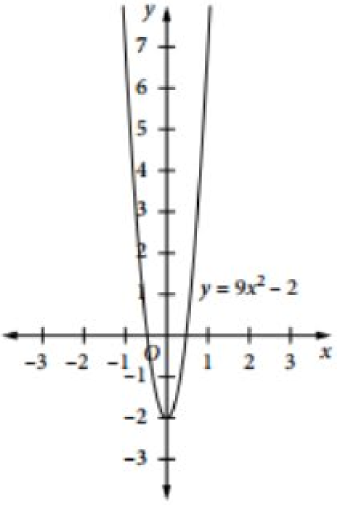




(b)

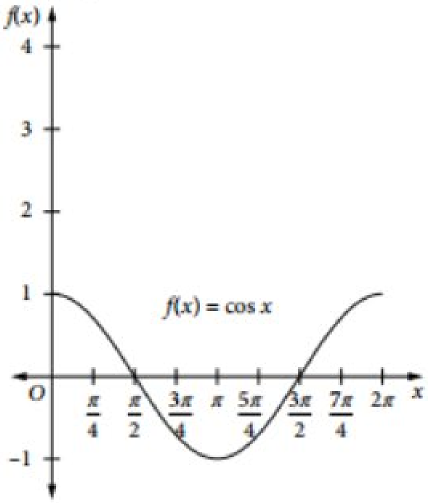


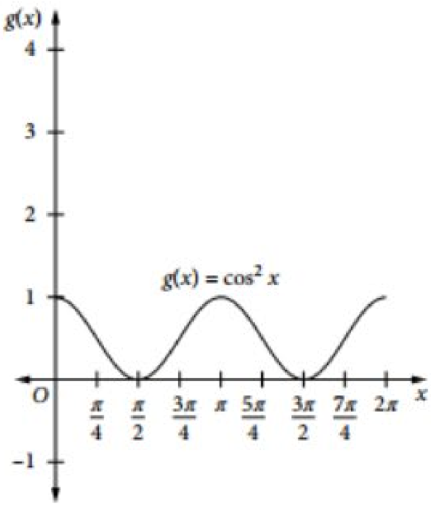


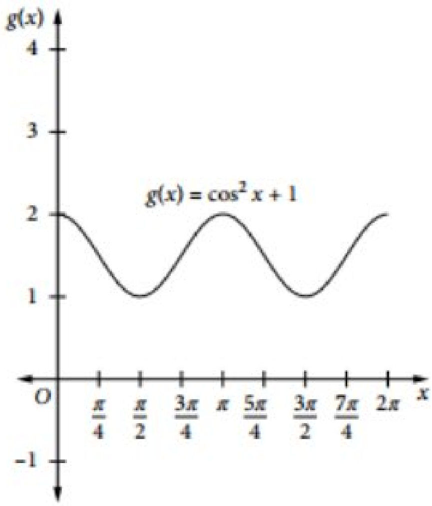


(c) The graphs are the same.

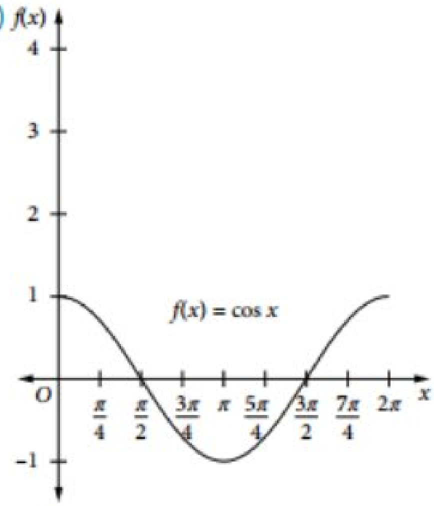
12 (a)

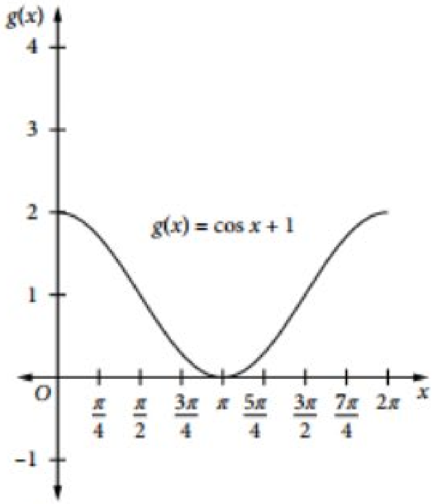


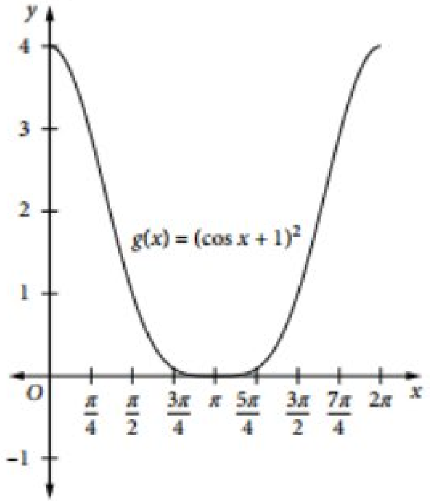




(b)

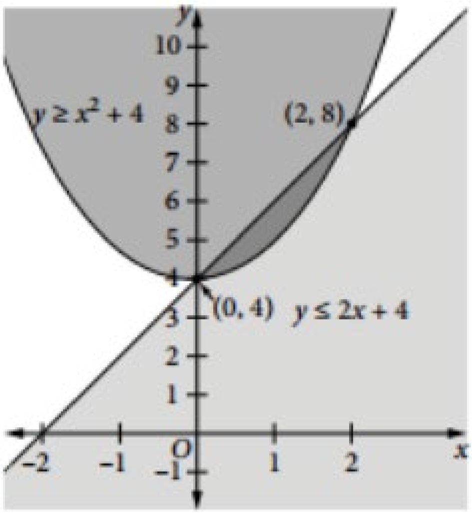






(c) The final graphs in each part are completely different due to the time the squaring is done.

13 (a)



(b) $0 \leq x \leq 2$