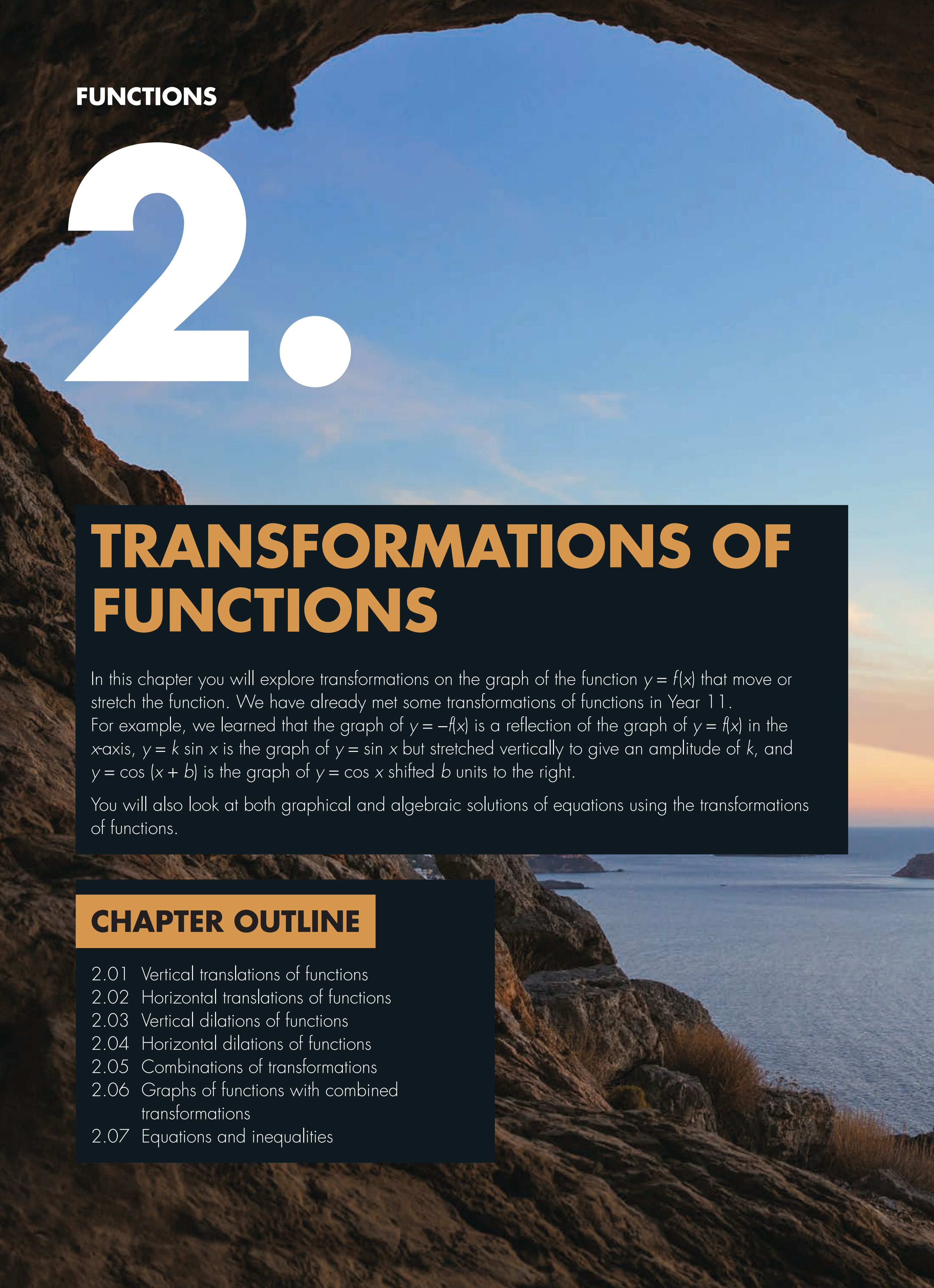


FUNCTIONS

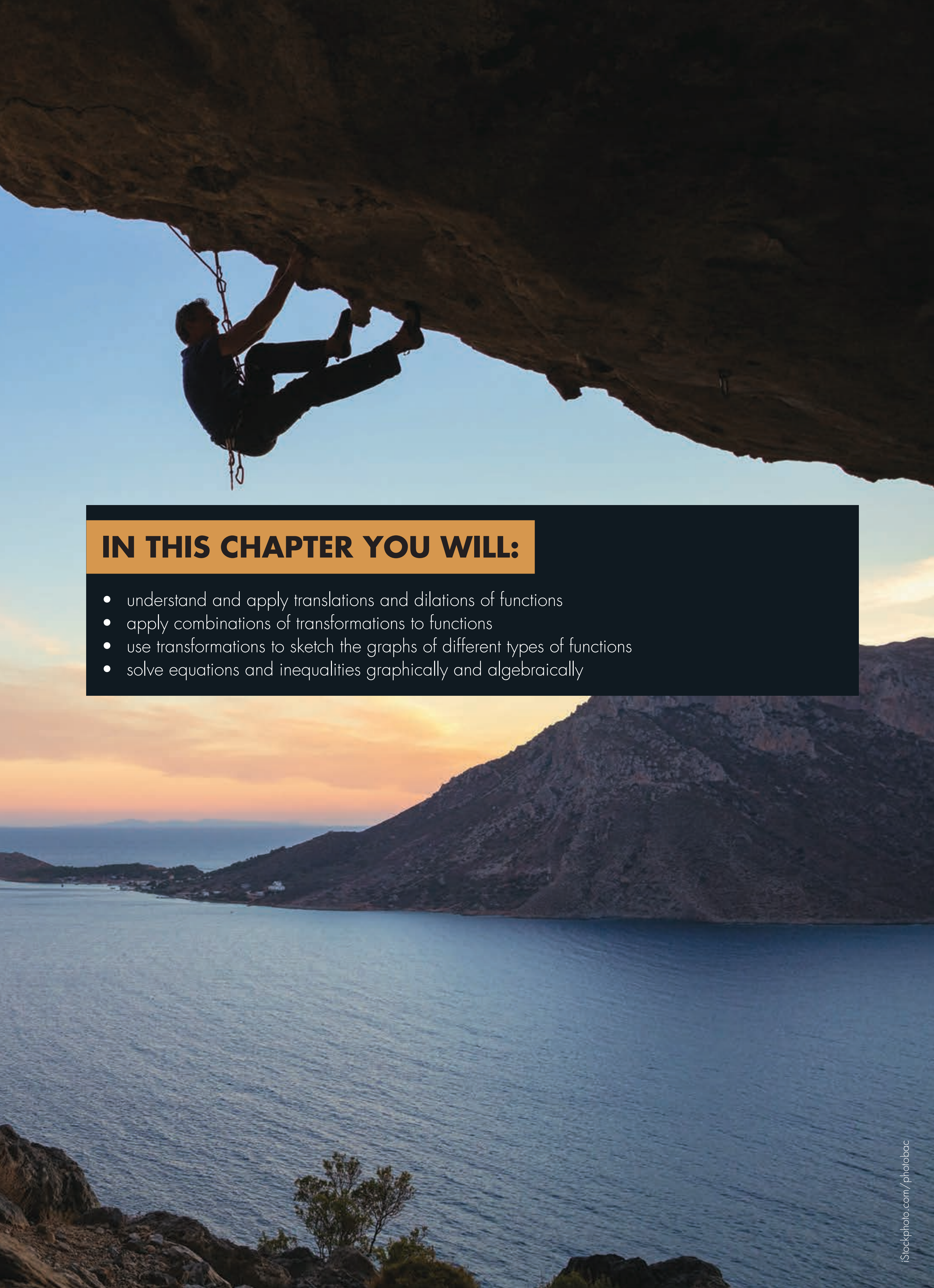
2.

TRANSFORMATIONS OF FUNCTIONS

In this chapter you will explore transformations on the graph of the function $y = f(x)$ that move or stretch the function. We have already met some transformations of functions in Year 11. For example, we learned that the graph of $y = -f(x)$ is a reflection of the graph of $y = f(x)$ in the x-axis, $y = k \sin x$ is the graph of $y = \sin x$ but stretched vertically to give an amplitude of k, and $y = \cos (x + b)$ is the graph of $y = \cos x$ shifted b units to the right.

You will also look at both graphical and algebraic solutions of equations using the transformations of functions.

CHAPTER OUTLINE



IN THIS CHAPTER YOU WILL:

- understand and apply translations and dilations of functions
- apply combinations of transformations to functions
- use transformations to sketch the graphs of different types of functions
- solve equations and inequalities graphically and algebraically

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TERMINOLOGY

dilation: The process of stretching or compressing the graph of a function horizontally or vertically.

parameter: a constant in the equation of a function that determines the properties of that function and its graph; for example, the parameters for $y = mx + c$ are m (gradient) and c (y-intercept).

scale factor: The value of k by which the graph of a function is dilated.

transformation: A general name for the process of changing the graph of a function by moving, reflecting or stretching it.

translation: The process of shifting the graph of a function horizontally and/or vertically without changing its size or shape.

2.01 Vertical translations of functions

INVESTIGATION

VERTICAL TRANSLATIONS

Some graphics calculators or graphing software use a dynamic feature to show how a constant c (a **parameter**) changes the graph of a function.

Use dynamic geometry software to explore the effect of c on each graph below. If you don't have dynamic software, substitute different values for c into the equation. Use positive and negative values, integers and fractions.

1 $f(x) = x + c$

2 $f(x) = x^2 + c$

3 $f(x) = x^3 + c$

4 $f(x) = x^4 + c$

5 $f(x) = e^x + c$

6 $f(x) = \ln x + c$

7 $f(x) = \frac{1}{x} + c$

8 $f(x) = |x| + c$

How does the value of c transform the graph? What is the difference between positive and negative values of c?

Notice that c shifts the graph up and down without changing its size or shape. We call this a **vertical translation** (a shift along the y-axis).

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Vertical translation

For the function $y = f(x)$:

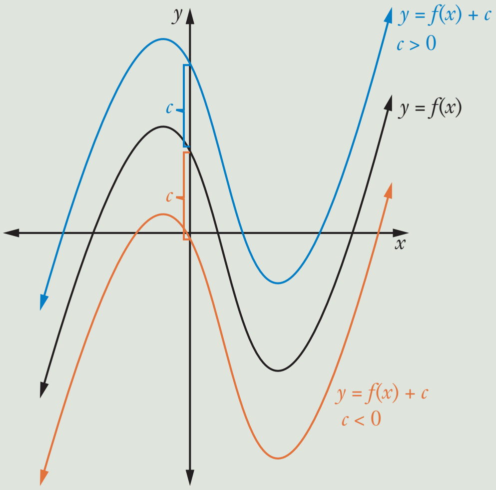
$y = f(x) + c$ translates the graph vertically (along the y-axis).

If $c > 0$, the graph is translated upwards by c units.

If $c < 0$, the graph is translated downwards.

A vertical translation changes the y values of the function.

In Year 11, we learned that $y = \sin x + c$ is the graph of $y = \sin x$ shifted up c units.



EXAMPLE 1

a Explain how the graph of $y = x^2 + 2$ is related to the graph of $y = x^2$.

b If the graph of the function $y = x^2 + 7x + 1$ is translated 4 units down, find the equation of the transformed function.

c The point $P(3, -2)$ lies on the function $y = f(x)$. Find the transformed point (the image of P) if the function is translated:

i 6 units down **ii** 8 units up

Solution

a The graph of $y = x^2 + 2$ is a vertical translation 2 units up from the original (parent) function $y = x^2$.

b For a vertical translation 4 units down:

$y = f(x) + c$ where $c = -4$

$y = x^2 + 7x + 1 - 4$

$\quad = x^2 + 7x - 3$

The equation of the transformed function is $y = x^2 + 7x - 3$

c **i** $P(3, -2)$ is translated 6 units down, so subtract 6 from the y value.

The transformed point is $(3, -2 - 6) \equiv (3, -8)$.

For points, we use '≡' (identical to) rather than '='.

ii $P(3, -2)$ is translated 8 units up, so add 8 to the y value.

The transformed point is $(3, -2 + 8) \equiv (3, 6)$.

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EXAMPLE 2

a Sketch the graph of $y = x^3 - 3$.

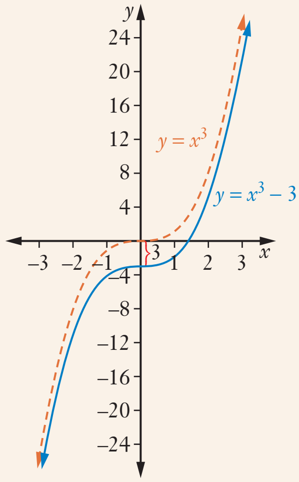
b **i** State the relationship of $y = \frac{1}{x} - 2$ to $y = \frac{1}{x}$.

ii State the domain and range of $y = \frac{1}{x} - 2$

iii Sketch the graph of $y = \frac{1}{x} - 2$.

Solution

a A vertical translation of –3 units shifts the function $y = x^3$ down to the graph of $y = x^3 - 3$.
If you need to find some points on the graph of $y = x^3 - 3$ you could subtract 3 from y values of $y = x^3$.



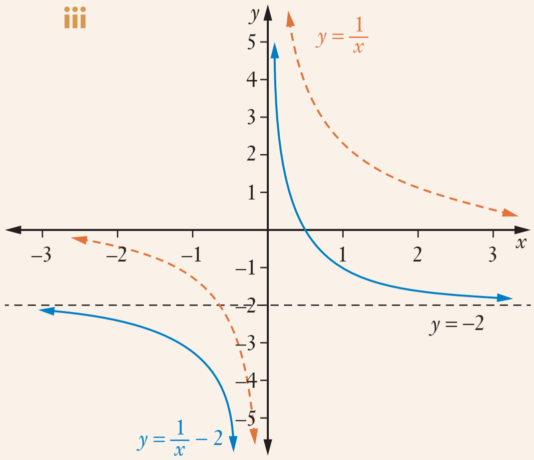
b **i** $y = \frac{1}{x} - 2$ is a vertical translation 2 units down of $y = \frac{1}{x}$.

ii Since $x \neq 0$, domain is $(-\infty, 0) \cup (0, \infty)$.

Since $\frac{1}{x} \neq 0$, $\frac{1}{x} - 2 \neq -2$
So range is $(-\infty, -2) \cup (-2, \infty)$.

Since the horizontal asymptote is at $y = -2$, we sketch it as a dotted line.

iii



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Exercise 2.01 Vertical translations of functions

1 Describe how each constant affects the graph of $y = x^2$.

a $y = x^2 + 3$ **b** $y = x^2 - 7$

c $y = x^2 - 1$ **d** $y = x^2 + 5$

2 Describe how each constant affects the graph of $y = x^3$.

a $y = x^3 + 1$ **b** $y = x^3 - 4$ **c** $y = x^3 + 8$

3 Describe how the graph of $y = \frac{1}{x}$ transforms to the graph of $y = \frac{1}{x} + 9$.

4 Find the equation of each translated function.

a $y = x^2$ is translated 3 units downwards

b $f(x) = 2^x$ is translated 8 units upwards

c $y = |x|$ is translated 1 unit upwards

d $y = x^3$ is translated 4 units downwards

e $f(x) = \log x$ is translated 3 units upwards

f $y = \frac{2}{x}$ is translated 7 units downwards

5 Describe the relationship between the graph of $f(x) = x^4$ and:

a $f(x) = x^4 - 1$ **b** $f(x) = x^4 + 6$

6 Find the equation of the transformed function if:

a $y = 2x^3 + 3$ is translated:

i 5 units down **ii** 3 units up

b $y = |x| - 4$ is translated:

i 1 unit up **ii** 2 units down

c $y = e^x + 2$ is translated:

i 1 unit down **ii** 3 units up

d $y = \log_e x - 1$ is translated:

i 11 units up **ii** 7 units down

7 If $P = (1, -3)$ lies on the function $y = f(x)$, find the transformed (image) point of P if the function is translated:

a 2 units up **b** 6 units down **c** m units up

8 Find the original point P on the function $y = f(x)$ if the coordinates of its transformed image are $(-1, 2)$ when the function is translated:

a 1 unit up **b** 3 units down

9 Sketch each set of functions on the same number plane.

a $y = x^2$, $y = x^2 + 2$ and $y = x^2 - 3$

b $y = 3^x$ and $y = 3^x - 4$

c $y = |x|$ and $y = |x| - 3$

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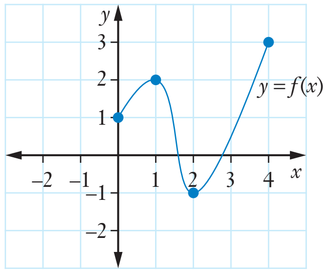
10 a Describe the transformation of $y = \frac{1}{x}$ into $y = \frac{1}{x} + 1$.

b Sketch the graph of $y = \frac{1}{x} + 1$.

11 The graph shows $y = f(x)$. Sketch the graph of:

a $y = f(x) - 1$

b $y = f(x) + 2$



12 a Show that $\frac{3x+1}{x} = \frac{1}{x} + 3$.

b Hence or otherwise, sketch the graph of $y = \frac{3x+1}{x}$.

2.02 Horizontal translations of functions

WS Translations of functions

INVESTIGATION

HORIZONTAL TRANSLATIONS

Use a graphics calculator or graphing software to explore the affect of parameter b on each graph below. If you don't have dynamic software, substitute different values for b into the equation. Use positive and negative values, integers and fractions for b.

WS Graphing translations of functions

1 $f(x) = (x + b)^2$

2 $f(x) = (x + b)^3$

3 $f(x) = (x + b)^4$

4 $f(x) = e^{(x + b)}$

5 $f(x) = \ln(x + b)$

6 $f(x) = \frac{1}{x + b}$

7 $f(x) = |x + b|$

How does the graph change as the value of b changes?

What is the difference between positive and negative values of b?

Notice that the parameter shifts the graph to the left or right without changing its size or shape. We call this a **horizontal translation** (it shifts the function along the x-axis).

For a horizontal translation the shift is in the opposite direction from the sign of b.

To understand why this happens, we change the subject of the equation to x since the translation is a shift along the x-axis. For example:

$$y = (x + 5)^3$$
$$\sqrt[3]{y} = x + 5$$
$$\sqrt[3]{y} - 5 = x$$

This is a shift of 5 units to the left.

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Horizontal translations

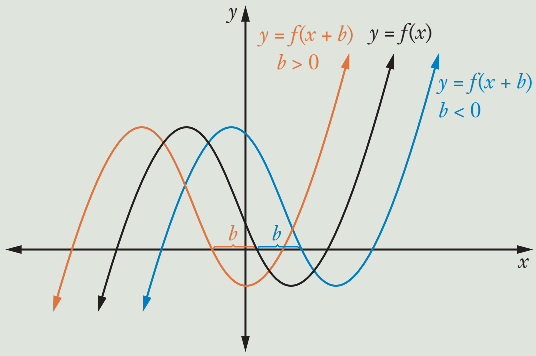
For the function $y = f(x)$:

$y = f(x + b)$ translates the graph horizontally (along the x-axis).

If $b > 0$, the graph is translated to the left by b units.

If $b < 0$, the graph is translated to the right.

A horizontal translation changes the x values of the function.



In Year 11, we learned that $y = \tan(x + b)$ is the graph of $y = \tan x$ shifted left b units.

EXAMPLE 3

a What is the relationship of $f(x) = \log_2(x + 3)$ to $f(x) = \log_2 x$?

b If the graph $y = (x - 4)^3$ is translated 7 units to the right, find the equation of the transformed function.

c The point $P(2, 5)$ lies on the function $y = f(x)$. Find the corresponding (image) point of P given a horizontal translation with $b = 1$.

d The point Q $(3, -4)$ on the graph of $y = f(x - 2)$ is the image of point $P(x, y)$ on $y = f(x)$. Find the coordinates of P.

Solution

a $f(x) = \log_2(x + 3)$ is a horizontal translation 3 units to the left from the parent function $f(x) = \log_2 x$.

b If $y = (x - 4)^3$ is translated 7 units to the right:

$y = f(x + b)$ where $b = -7$

$y = (x - 4 - 7)^3 = (x - 11)^3$

So the equation of the transformed function is $y = (x - 11)^3$

c $y = f(x + b)$ describes a horizontal translation (along the x-axis).

When $b = 1$, x values shift 1 unit to the left.

Image of $P \equiv (2 - 1, 5) \equiv (1, 5)$

d $y = f(x - 2)$ is a horizontal translation 2 units to the right of $y = f(x)$.

So (x, y) becomes $(x + 2, y)$

But $Q(3, -4)$ is the image of $P(x, y)$

So $(x + 2, y) \equiv (3, -4)$

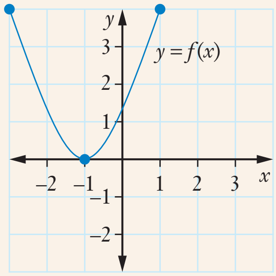
So $x + 2 = 3, y = -4$

$x = 1, y = -4$

So $P \equiv (1, -4)$

EXAMPLE 4

a The graph of $y = f(x)$ shown is transformed into $y = f(x + b)$. Sketch the transformed graph if $b = -3$.



b Sketch the graph of:

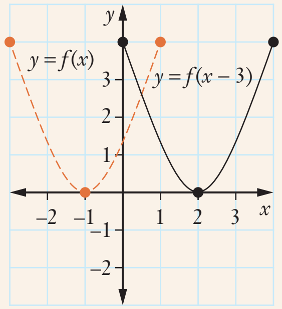
i $y = |x + 3|$

ii $y = \dfrac{1}{x - 2}$

Solution

a The graph $y = f(x + b)$ where $b = -3$ describes a horizontal translation of 3 units to the right.

The transformed graph is 3 units to the right of the original function.



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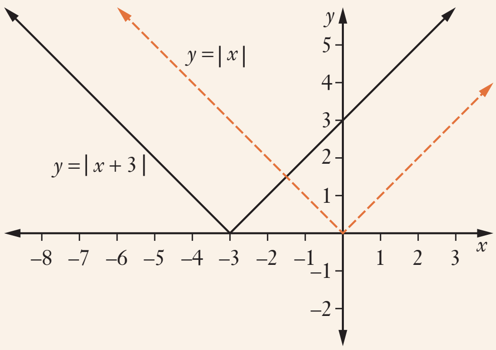
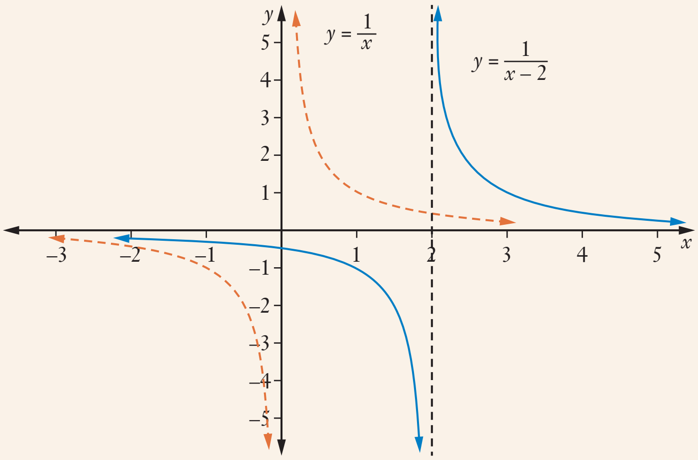
b **i** The function $y = |x + 3|$ is in the form $y = f(x + b)$ where $b = 3$.

Since $b > 0$, $y = |x|$ is shifted 3 units to the left.



If you need to find some points on the graph of $y = |x + 3|$ you could subtract 3 from x values of $y = |x|$.

ii $y = \dfrac{1}{x-2}$ is in the form $y = f(x + b)$ where $b = -2$.
Since $b < 0$, $y = \dfrac{1}{x}$ is shifted 2 units to the right.



Exercise 2.02 Horizontal translations of functions

1 Describe how each constant affects the graph of $y = x^2$.

a $y = (x - 4)^2$ **b** $y = (x + 2)^2$

2 Describe how each constant affects the graph of $y = x^3$.

a $y = (x - 5)^3$ **b** $y = (x + 3)^3$

3 Find the equation of each translated graph.

a $y = x^2$ translated 3 units to the left **b** $f(x) = 2^x$ translated 8 units to the right

c $y = |x|$ translated 1 unit to the left **d** $y = x^3$ translated 4 units to the right

e $f(x) = \log x$ translated 3 units left

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4 Describe how $y = \dfrac{1}{x}$ transforms to $y = \dfrac{1}{x-3}$.

5 Describe the relationship between $f(x) = x^4$ and:

a $f(x) = (x+2)^4$ **b** $f(x) = (x-5)^4$

6 Find the equation if:

a $y = -x^2$ is translated

i 4 units to the left **ii** 8 units to the right

b $y = |x|$ is translated

i 3 units to the right **ii** 4 units to the left

c $y = e^{x+2}$ is translated

i 4 units to the left **ii** 7 units to the right

d $y = \log_2(x-3)$ is translated

i 2 units to the right **ii** 3 units to the left

7 If $P = (1, -3)$ lies on the function $y = f(x)$, find the image point of P if the function is transformed to $y = f(x+b)$ where:

a $b = -4$ **b** $b = 9$ **c** $b = t$

8 Find the original point on the function $y = f(x)$ if the coordinates of its image are $(-1, 2)$ when the function is translated:

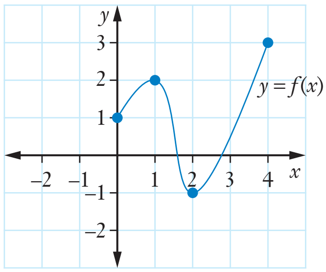
a 4 units to the left **b** 8 units to the right

9 Sketch on the same number plane:

a $y = x^3$ and $y = (x+1)^3$ **b** $f(x) = \ln x$ and $f(x) = \ln(x+2)$

10 The graph shown is $y = f(x)$. Sketch the graph of:

a $y = f(x-1)$ **b** $y = f(x+3)$



11 Find the equation of the transformed function if $f(x) = x^5$ is translated:

a 5 units down **b** 3 units to the right

c 2 units up **d** 7 units to the left

12 The point $P(3, -2)$ is the image of a point on $y = f(x)$ after it has been translated 4 units to the left. Find the original point.

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2.03 Vertical dilations of functions

A **dilation** stretches or compresses a function, changing its size and shape.

INVESTIGATION

VERTICAL DILATION

Explore the effect of parameter k on each graph below. If you don't have dynamic software, substitute different values for k into the equation. Use positive and negative values, integers and fractions for k.

1 $f(x) = kx$ **2** $f(x) = kx^2$ **3** $f(x) = kx^3$

4 $f(x) = kx^4$ **5** $f(x) = ke^x$ **6** $f(x) = k \ln x$

7 $f(x) = k\left(\frac{1}{x}\right)$ **8** $f(x) = k|x|$

How does the graph change as the value of k changes?

What is the difference between positive and negative values of k?

Notice that k stretches the graph up and down along the y-axis and changes its shape. We call this **vertical dilation**. The value of the parameter k controls the amount of stretching (expanding) or shrinking (compressing).

We call k the **scale factor**.

Vertical dilations

For the curve $y = f(x)$:

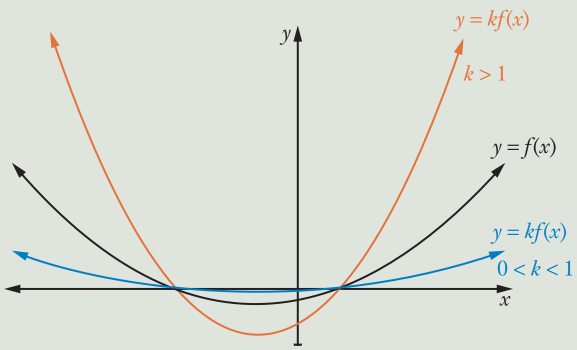
$y = kf(x)$ dilates the curve vertically (along the y-axis) by a scale factor of k.

If $k > 1$, the graph is stretched, or expanded.

If $0 < k < 1$, the graph is shrunk, or compressed.

A vertical dilation changes the y values of the function.

In Year 11, we learned that $y = k \cos x$ is the graph of $y = \cos x$ stretched vertically to give an amplitude of k.



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EXAMPLE 5

a The function $y = x^7$ is dilated vertically by a factor of 3. Find the equation of the transformed function.

b Describe how the function $f(x) = \frac{\log_2 x}{2}$ is related to the function $f(x) = \log_2 x$.

c Find the scale factor of each dilation of a function and state whether the dilation stretches or compresses the graph.

i $y = 7x^2$ **ii** $y = \frac{e^x}{5}$

Solution

a If a function $y = f(x)$ has a vertical dilation with factor k, the equation of its transformed function is $y = kf(x)$.

So if the function $y = x^7$ has a vertical dilation with factor 3, the equation of the transformed function is $y = 3x^7$.

Since $k > 1$, the function is stretched vertically.

b
$$\begin{aligned} f(x) &= \frac{\log_2 x}{2} \\ &= \frac{1}{2}\log_2 x \end{aligned}$$

So the function is in the form $y = kf(x)$ where $k = \frac{1}{2}$.

Since $0 < k < 1$, the function is compressed vertically.

So $f(x) = \frac{\log_2 x}{2}$ is the result of $f(x) = \log_2 x$ being dilated (compressed) vertically by a scale factor of $\frac{1}{2}$.

c The function $y = kf(x)$ has scale factor k.

i $y = 7x^2$ has scale factor 7 (stretched)

ii
$$\begin{aligned} y &= \frac{e^x}{5} \\ &= \frac{1}{5}e^x \end{aligned}$$

Scale factor is $\frac{1}{5}$ (compressed).

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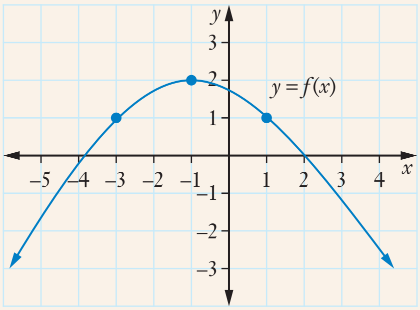
EXAMPLE 6

a The point $N = (-1, 8)$ lies on the function $y = f(x)$. Find the image of N on the function $y = kf(x)$ when:

i $k = 5$ **ii** $k = \frac{1}{2}$

b A function $y = f(x)$ is transformed to $y = kf(x)$. If the image of point A on the transformed function is $(-6, 12)$, find the coordinates of A when $k = 3$.

c The graph shown is $y = f(x)$. Sketch the graph of $y = 2f(x)$.



d Sketch the graphs of $y = x^2$ and $y = \frac{x^2}{2}$ on the same set of axes.

Solution

a $y = kf(x)$ describes a vertical dilation (along the y-axis).

So the y values of the parent function will change.

i When $k = 5$: y values are multiplied by a factor of 5.

Image of $N \equiv (-1, 8 \times 5) \equiv (-1, 40)$

ii When $k = \frac{1}{2}$: y values will be multiplied by a factor of $\frac{1}{2}$ (or divided by 2).

Image of $N \equiv \left(-1, 8 \times \frac{1}{2}\right) \equiv (-1, 4)$

b When $k = 3$, (x, y) becomes $(x, 3y)$.

$(x, 3y) \equiv (-6, 12)$

$x = -6$

$3y = 12$

$y = 4$

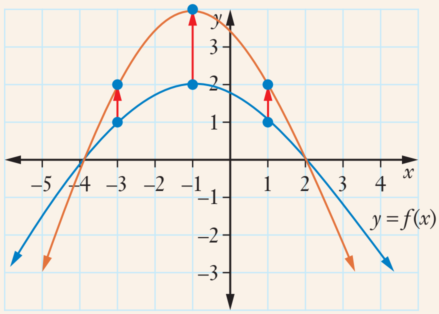
So $A \equiv (-6, 4)$

c The graph of $y = 2f(x)$ is a vertical dilation of $y = f(x)$ with factor 2.

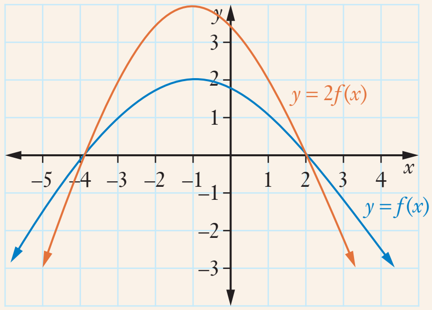
So each y value is doubled and the graph is twice as high as the original graph. For example:

$y = 1$ becomes $y = 2$

$y = 2$ becomes $y = 4$



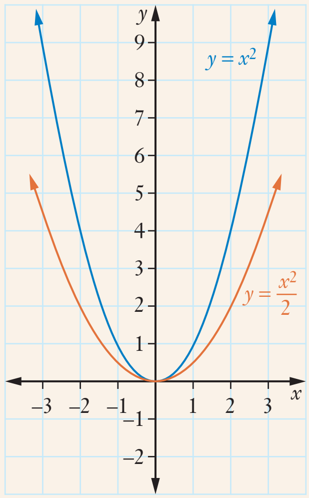
The transformed graph is still a parabola. However it is higher (stretched) and narrower than the original graph.



d $y = \dfrac{x^2}{2}$ is a vertical dilation of $y = x^2$ with scale factor $\dfrac{1}{2}$.

This halves the y values.

$(-3, 9)$	becomes	$\left(-3, 4\frac{1}{2}\right)$
$(-2, 4)$	becomes	$(-2, 2)$
$(-1, 1)$	becomes	$\left(-1, \frac{1}{2}\right)$
$(0, 0)$	becomes	$(0, 0)$
$(1, 1)$	becomes	$\left(-1, \frac{1}{2}\right)$
$(2, 4)$	becomes	$(2, 2)$
$(3, 9)$	becomes	$\left(3, 4\frac{1}{2}\right)$



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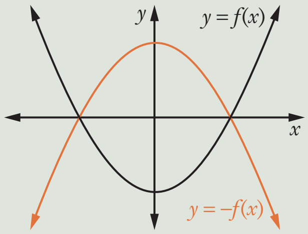
Reflections in the x-axis

You studied reflections in Year 11 in Chapter 7, *Further functions*.

Reflections in the x-axis

$y = -f(x)$ is a reflection of the curve $y = f(x)$ in the x-axis.

This is also a vertical dilation with scale factor $k = -1$.



EXAMPLE 7

a Point $P(2, 4)$ is on the function $y = f(x)$. Find the image of P on the function $y = -f(x)$.

b Sketch the vertical dilation of $f(x) = \frac{1}{x}$ with scale factor -1.

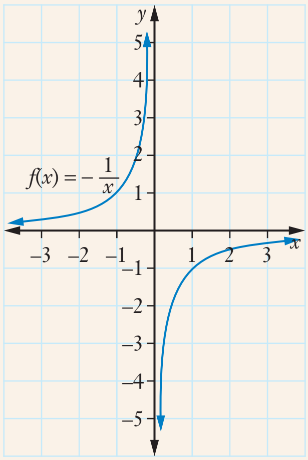
Solution

a The function $y = -f(x)$ is a reflection in the x-axis.

The y values are multiplied by -1.

Image of $P \equiv (2, 4 \times [-1]) \equiv (2, -4)$

b A vertical stretch with scale factor -1 is a reflection of $f(x) = \frac{1}{x}$ in the x-axis.



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Exercise 2.03 Vertical dilations of functions

1 Describe how the constant affects each transformed graph, given the parent function, and state the scale factor.

a $y = x$

i $y = 6x$ **ii** $y = \frac{x}{2}$ **iii** $y = -x$

b $y = x^2$

i $y = 2x^2$ **ii** $y = \frac{x^2}{6}$ **iii** $y = -x^2$

c $y = x^3$

i $y = 4x^3$ **ii** $y = \frac{x^3}{7}$ **iii** $y = \frac{4x^3}{3}$

d $y = x^4$

i $y = 9x^4$ **ii** $y = \frac{x^4}{3}$ **iii** $y = \frac{3x^4}{8}$

e $y = |x|$

i $y = 5|x|$ **ii** $y = \frac{|x|}{8}$ **iii** $y = -|x|$

f $f(x) = \log x$

i $f(x) = 9 \log x$ **ii** $f(x) = -\log x$ **iii** $f(x) = \frac{2\log x}{5}$

2 Find the equation of each transformed graph and state its domain and range.

a $y = x^2$ dilated vertically with a scale factor of 6

b $y = \ln x$ dilated vertically with a scale factor of $\frac{1}{4}$

c $f(x) = |x|$ reflected in the x-axis

d $f(x) = e^x$ dilated vertically with a scale factor of 4

e $y = \frac{1}{x}$ dilated vertically with a scale factor of 7

3 Find the equation of each transformed function after the vertical dilation given.

a $y = 3^x$ with scale factor 5 **b** $f(x) = x^2$ with scale factor $\frac{1}{3}$

c $y = x^3$ with scale factor -1 **d** $y = \frac{1}{x}$ with scale factor $\frac{1}{2}$

e $y = |x|$ with scale factor $\frac{2}{3}$

4 Point $M = (3, 6)$ lies on the graph of $y = f(x)$. Find the coordinates of the image of M when $f(x)$ is:

a dilated vertically with a factor of 4 **b** reflected in the x-axis

c dilated vertically with a factor of 12 **d** dilated vertically with a factor of $\frac{5}{6}$

5 The coordinates of the image of $X(x, y)$ are (4, 12) when $y = f(x)$ is vertically dilated. Find the coordinates of X if the scale factor is:

a 3 **b** 2 **c** $\frac{1}{3}$ **d** $\frac{3}{4}$ **e** -1

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6 Sketch each pair of functions on the same set of axes.

a $f(x) = \log_2 x$ and $f(x) = 2\log_2 x$

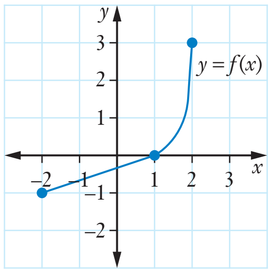
b $y = 3^x$ and $y = 2 \cdot 3^x$

c $y = \dfrac{1}{x}$ and $y = \dfrac{3}{x}$

d $y = |x|$ and $y = 2|x|$

e $y = x^3$ and $y = -x^3$

7 Points on a function $y = f(x)$ are shown on the graph. Sketch the graph of the transformed function showing the image points, given a vertical stretch with factor:



a 3

b $\dfrac{1}{2}$

c −1

8 Sketch the graph of $y = 2\sqrt{1-x^2}$.

2.04 Horizontal dilations of functions

WS Dilations of functions

WS Advanced graphs

INVESTIGATION

HORIZONTAL DILATIONS

Use dynamic geometry software to explore the affect of parameter a on each graph below. If you don't have dynamic software, substitute different values for a into the equation. Use positive and negative values, integers and fractions for a.

1 $f(x) = ax$

2 $f(x) = (ax)^2$

3 $f(x) = (ax)^3$

4 $f(x) = (ax)^4$

5 $f(x) = e^{ax}$

6 $f(x) = \ln ax$

7 $f(x) = \dfrac{1}{ax}$

8 $f(x) = |ax|$

How does a transform the graph as the value of a changes?

What is the difference between positive and negative values of a?

Notice that with **horizontal dilations**, the higher the value of a, the more the graph is compressed along the x-axis from left and right. This is inverse variation and the scale factor for horizontal dilations is $\dfrac{1}{a}$.

This is because horizontal dilation affects the x values of the function. To see this, we change the subject of the function to x. For example:

$$y = (3x)^3$$

$$\sqrt[3]{y} = 3x$$

$$\frac{\sqrt[3]{y}}{3} = x$$

$$\text{or } x = \frac{1}{3}\sqrt[3]{y}$$

In Year 11, we learned that $y = \sin ax$ is the graph of $y = \sin x$ compressed horizontally to give a period of $\dfrac{2\pi}{a}$.

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This shows a scale factor of $\frac{1}{3}$.

Like horizontal translations, a horizontal stretch works the opposite way to what you would expect, because the equation is in the form $y = f(x)$ rather than $x = f(y)$.

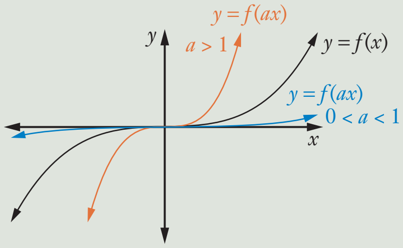
Horizontal dilations

For the curve $y = f(x)$:

$y = f(ax)$ stretches the curve horizontally (along the x-axis) by a scale factor of $\frac{1}{a}$.

If $a > 1$, the graph is compressed.

If $0 < a < 1$, the graph is stretched.



EXAMPLE 8

a Describe how the function $f(x) = x^3$ is related to the function $f(x) = (4x)^3$.

b The function $y = \ln x$ is dilated horizontally by a scale factor of 2. Find the equation of the transformed function.

c Find the scale factor of each function and state whether it stretches or compresses the graph.

i $y = e^{3x}$ **ii** $f(x) = \left|\frac{x}{4}\right|$

Solution

a The function $y = f(ax)$ is a horizontal dilation of $y = f(x)$ with scale factor $\frac{1}{a}$.

So the function $f(x) = (4x)^3$ is a horizontal dilation of $f(x) = x^3$ with scale factor $\frac{1}{4}$.

b If $y = \ln x$ is dilated horizontally by a scale factor of 2:

$$\frac{1}{a} = 2$$

$$a = \frac{1}{2}$$

So $y = \ln\left(\frac{1}{2}x\right)$ or $y = \ln\frac{x}{2}$

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c **i** $y = e^{3x}$ is in the form $y = f(ax)$ where $f(x) = e^x$.

This is a horizontal dilation with $a = 3$.

Scale factor $= \frac{1}{a} = \frac{1}{3}$ (stretched)

ii $f(x) = \left|\frac{x}{4}\right|$ can be written as $f(x) = \left|\frac{1}{4}x\right|$.

The function is in the form $y = f(ax)$ where $f(x) = |x|$.

This is a horizontal dilation with $a = \frac{1}{4}$.

$$\begin{aligned}\text{Scale factor} &= \frac{1}{a} \\ &= \frac{1}{\frac{1}{4}} \\ &= 4 \text{ (compressed)}\end{aligned}$$

EXAMPLE 9

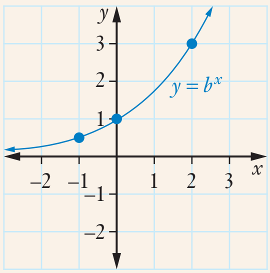
a The points $P(-3, 4)$ and $Q(9, 0)$ lie on the function $y = f(x)$. Find the coordinates of the images of P and Q for the function $y = f(ax)$ when:

i $a = 3$ **ii** $a = \frac{1}{5}$

b When the function $y = f(x)$ is transformed to $y = f(ax)$, the coordinates of the image of $N(x, y)$ are $(16, -5)$. Find the coordinates of N when:

i $a = 4$ **ii** $a = \frac{1}{2}$

c The graph of $y = b^x$ shown is transformed to $y = b^{2x}$. Sketch the graph of the transformed function.



d State the scale factor if the graph $y = |x|$ is transformed to $y = \left|\frac{x}{2}\right|$ and sketch both graphs on the same set of axes.

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Solution

a The function $y = f(ax)$ is a horizontal stretch of $y = f(x)$ with scale factor $\frac{1}{a}$.

i When $a = 3$, scale factor is $\frac{1}{3}$.

All x values are multiplied by $\frac{1}{3}$ (divided by 3).

$$\text{Image of } P \equiv \left(-3 \times \frac{1}{3},\ 4\right) \equiv (-1, 4)$$

$$\text{Image of } Q \equiv \left(9 \times \frac{1}{3},\ 0\right) \equiv (3, 0)$$

ii When $a = \frac{1}{5}$, scale factor is $\frac{1}{\frac{1}{5}}$ or 5.

All x values are multiplied by 5.

$$\text{Image of } P \equiv (-3 \times 5, 4) \equiv (-15, 4)$$

$$\text{Image of } Q \equiv (9 \times 5, 0) \equiv (45, 0)$$

b We multiply all x values by scale factor $\frac{1}{a}$.

i When $a = 4$, scale factor is $\frac{1}{4}$

So (x, y) becomes $\left(x \times \frac{1}{4}, y\right) = \left(\frac{x}{4}, y\right)$

$$\left(\frac{x}{4}, y\right) \equiv (16, -5)$$

$$\frac{x}{4} = 16$$

$$x = 64$$

$$y = -5$$

So $N \equiv (64, -5)$

ii When $a = \frac{1}{2}$, scale factor is $\frac{1}{\frac{1}{2}}$ or 2.

So (x, y) becomes $(x \times 2, y) \equiv (2x, y)$

$$(2x, y) \equiv (16, -5)$$

$$2x = 16$$

$$x = 8$$

$$y = -5$$

So $N \equiv (8, -5)$

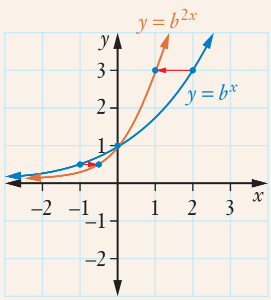
c The graph of $y = b^{2x}$ describes a horizontal dilation of $y = b^x$ with scale factor $\frac{1}{2}$.

So we halve the x values.

$x = -1$ becomes $x = -\frac{1}{2}$

$x = 0$ becomes $x = 0$

$x = 2$ becomes $x = 1$



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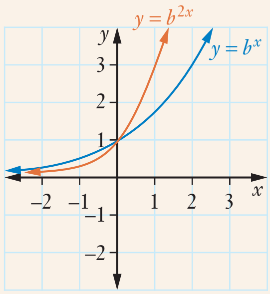
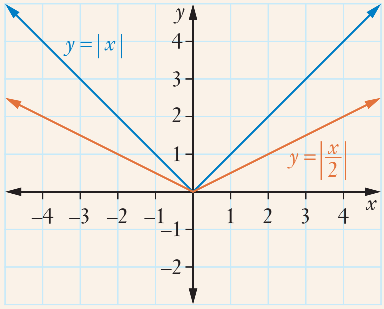
The transformed function is still in the shape of an exponential function, but it has changed shape and size.



d The graph $y = \left|\frac{x}{2}\right|$ is a horizontal dilation of $y = |x|$ with a scale factor $\frac{1}{\frac{1}{2}}$ or 2.

We double the x values.

(−3, 3)	becomes	(−6, 3)
(−2, 2)	becomes	(−4, 2)
(−1, 1)	becomes	(−2, 1)
(0, 0)	becomes	(0, 0)
(1, 1)	becomes	(2, 1)
(2, 2)	becomes	(4, 2)
(3, 3)	becomes	(6, 3)



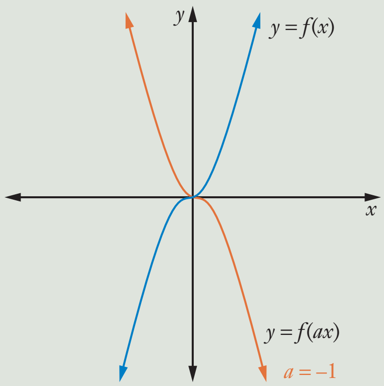
Reflections in the *y*-axis

You studied reflections in the y-axis in Year 11 in Chapter 7, *Further functions*.

Reflections in the *y*-axis

$y = f(-x)$ is a reflection of the curve $y = f(x)$ in the y-axis.

This is a horizontal stretch with scale factor $a = \frac{1}{-1} = -1$.



Notice that for even functions $y = f(x) = f(-x)$.

Even functions are already symmetrical about the y-axis so the reflected graph is the same as the original graph.

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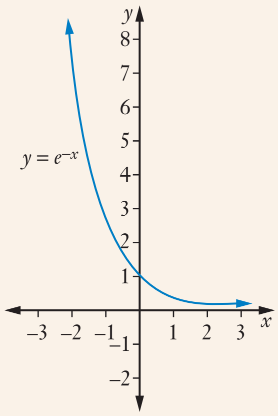
EXAMPLE 10

Sketch the graph of the horizontal dilation of $y = e^x$ with scale factor -1.

Solution

The horizontal dilation with scale factor -1 is a reflection of $y = e^x$ in the y-axis.



Exercise 2.04 Horizontal dilations of functions

1 Describe the transformation that the constant makes on $f(x) = x^4$ and state the scale factor.

a $f(x) = (8x)^4$ **b** $f(x) = \left(\frac{x}{5}\right)^4$

c $f(x) = \left(\frac{3x}{7}\right)^4$ **d** $f(x) = (-x)^4$

2 Describe whether the constant describes a horizontal or vertical dilation and state the scale factor.

a $y = x^2$

i $y = (2x)^2$ **ii** $y = (5x)^2$ **iii** $y = \left(\frac{x}{3}\right)^2$

b $y = x^3$

i $y = 4x^3$ **ii** $y = \left(\frac{x}{2}\right)^3$ **iii** $y = (-x)^3$

c $y = x^4$

i $y = (7x)^4$ **ii** $y = \frac{x^4}{8}$ **iii** $y = \left(\frac{3x}{4}\right)^4$

d $y = |x|$

i $y = |5x|$ **ii** $y = \left|\frac{x}{2}\right|$ **iii** $y = \left|\frac{3x}{5}\right|$

e $y = 5^x$

i $y = 5^{3x}$ **ii** $y = -5^x$ **iii** $y = 5^{\frac{x}{2}}$

f $f(x) = \log x$

i $f(x) = 8 \log x$ **ii** $f(x) = \log(-x)$ **iii** $f(x) = \log \frac{x}{7}$

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3 Find the equation of each transformed graph and state its domain and range.

a $f(x) = |x|$ is dilated horizontally with a scale factor of $\frac{1}{5}$

b $y = x^2$ is dilated horizontally with a scale factor of 3

c $y = x^3$ is reflected in the y-axis.

d $y = e^x$ is dilated vertically with a scale factor $\frac{1}{9}$

e $y = \log_4 x$ is reflected in the x-axis

4 Point X (–2, 7) lies on $y = f(x)$. Find the coordinates of the image of X on $y = f(ax)$ given:

a $a = 2$ **b** $a = -1$ **c** $a = \frac{1}{3}$

5 The function $y = f(x)$ is transformed into the function $y = f(ax)$. The coordinates of the image point of (x, y) on the original function are (–24, 1) on the transformed function. Find the values of (x, y) if:

a $a = 3$ **b** $a = 2$ **c** $a = \frac{1}{4}$

6 Sketch each pair of functions on the same set of axes:

a $f(x) = \ln x$ and $f(x) = \ln(2x)$

b $y = 2^x$ and $y = 2^{\frac{x}{3}}$

c $y = \frac{1}{x}$ and $y = \frac{1}{3x}$

d $y = |x|$ and $y = |2x|$

e $f(x) = x^2$ and $f(x) = (3x)^2$

f $y = \ln x$ and $y = \ln(-x)$

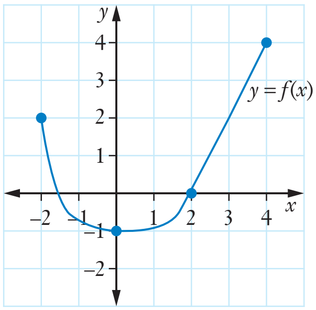
7 Sketch the graphs of $y = e^x$, $y = e^{2x}$ and $y = 2e^x$ on the same set of axes.

8 Explain why a reflection in the y-axis does not change the graph of:

a $y = x^2$ **b** $f(x) = |x|$

9 Sketch the graph of $y = f(ax)$ given the graph of $y = f(x)$ shown, when:

a $a = \frac{1}{2}$ **b** $a = 2$



WS
Combinations of transformations

2.05 Combinations of transformations

A function can have any combination of the different types of transformations acting on it.

Transformations

For the curve $y = f(x)$:

$y = f(x) + c$ translates the function vertically:

- up if $c > 0$
- down if $c < 0$

$y = f(x + b)$ translates the function horizontally:

- to the left if $b > 0$
- to the right if $b < 0$

$y = kf(x)$ dilates the function vertically with scale factor k:

- stretches if $k > 1$
- compresses if $0 < k < 1$
- reflects the function in the x-axis if $k = -1$

$y = f(ax)$ dilates the function horizontally with scale factor $\frac{1}{a}$:

- compresses if $a > 1$
- stretches if $0 < a < 1$
- reflects the function in the y-axis if $a = -1$

EXAMPLE 11

a Find the equation of the transformed function if $y = x^4$ is shifted 2 units down and 5 units to the left.

b Find the equation of the transformed function if $y = e^x$ is dilated vertically by a scale factor 3 and translated horizontally 2 units to the right.

Solution

a Starting with $y = x^4$:

A vertical translation 2 units down gives $y = x^4 - 2$.

A horizontal translation 5 units to the left gives $b = 5$.

So the equation becomes $y = (x + 5)^4 - 2$.

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Notice that we could do this the other way around:

A horizontal translation 5 units to the left gives $y = (x + 5)^4$.

A vertical translation 2 units down gives $y = (x + 5)^4 - 2$.

b Starting with $y = e^x$:

A vertical dilation of scale factor 3 gives $y = 3e^x$.

A horizontal translation 2 units to the right gives $b = -2$.

So the equation becomes $y = 3e^{x-2}$.

Notice that we could do this the other way around:

A horizontal translation 2 units to the right gives $y = e^{x-2}$.

A vertical dilation of scale factor 3 gives $y = 3e^{x-2}$.

When the transformations are *both* vertical or *both* horizontal, then the order is important.

EXAMPLE 12

a When the function $y = x^2$ is translated 3 units up (vertically) and vertically dilated by scale factor 4, the equation of the transformed function is $y = 4x^2 + 3$. Find the order in which the transformations were done.

b The equation of the transformed function is $y = (2x + 5)^3$ when the function $y = x^3$ is horizontally dilated by scale factor $\frac{1}{2}$ and translated 5 units (horizontally) to the left. In which order were the transformations done?

c The equation of the transformed function is $y = \ln [3(x - 2)]$ when the function $y = \ln x$ is horizontally dilated by scale factor $\frac{1}{3}$ and translated 2 units (horizontally) to the right. In which order were the transformations done?

Solution

a Starting with $y = x^2$:

A vertical translation 3 units up gives $y = x^2 + 3$.

A vertical dilation by scale factor 4 gives $y = 4(x^2 + 3)$.

This is not the equation of the transformed function.

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Try the other way around:

A vertical dilation by scale factor 4 gives $y = 4x^2$.

A vertical translation 3 units up gives $y = 4x^2 + 3$.

So the correct order is the vertical dilation, then the vertical translation.

b Starting with $y = x^3$:

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$, so the equation is $y = (2x)^3$.

A horizontal translation 5 units to the left gives $b = 5$ so $y = [2(x + 5)]^3$.

This is not the equation of the transformed function.

Try the other way around:

A horizontal translation 5 units to the left gives $y = (x + 5)^3$.

A horizontal dilation of scale factor $\frac{1}{2}$ gives $a = 2$, so the equation is $y = (2x + 5)^3$.

So the correct order is the horizontal translation, then the horizontal dilation.

c Starting with $y = \ln x$:

A horizontal dilation of scale factor $\frac{1}{3}$ gives $y = \ln (3x)$.

A horizontal translation 2 units to the right gives $y = \ln [3(x - 2)]$.

So the correct order is the horizontal dilation, then the horizontal translation.

Doing the horizontal dilation first gives $y = f(a(x + b))$, while doing the horizontal translation first gives $y = f(ax + b)$.

We can state the order we want to perform the transformations.

EXAMPLE 13

Find the equation of the function if $y = x^2$ is first horizontally dilated with scale factor $\frac{1}{2}$, then translated 3 units to the right.

Solution

A horizontal dilation with scale factor $\frac{1}{2}$ gives $a = 2$.

So $y = x^2$ becomes $y = (2x)^2$.

A horizontal translation 3 units to the right gives $b = -3$.

So $y = (2x)^2$ transforms to $y = [2(x - 3)]^2$.

Remember to put brackets around $x - 3$.

We can combine all the transformations into a single expression:

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Equation of a transformed function

$y = kf(a(x + b)) + c$ where a, b, c and k are constants is a transformation of $y = f(x)$:

- a horizontal dilation of scale factor $\dfrac{1}{a}$
- a horizontal translation of b
- a vertical dilation of k
- a vertical translation of c.

Order of transformations

For $y = kf(a(x + b)) + c$:

1 do horizontal dilation (a), then horizontal translation (b)

2 do vertical dilation (k), then vertical translation (c)

It doesn't matter whether you do horizontal or vertical transformations first.

Notice that the horizontal dilation and translation parameters a and b are inside the brackets (they change x values) and the vertical dilation and translation parameters k and c are outside the brackets (they change the y values).

EXAMPLE 14

a Describe the transformations of $y = e^x$ in the correct order to produce the transformed function $y = \dfrac{1}{2}e^{x+1} - 3$.

b Describe the transformations of $y = x^2$ in order that give the transformed function $y = 3(2x - 6)^2 + 1$.

c Find the equation of the transformed function if $y = f(x)$ undergoes a vertical dilation with factor 5, a horizontal dilation with factor −1, a translation 4 units to the right and 9 units down.

Solution

a For $y = \dfrac{1}{2}e^{x+1} - 3$:

Horizontal transformations (a and b): No dilation, $b = 1$ gives a translation 1 unit left.

Vertical transformations (k and c): dilation of scale factor $\dfrac{1}{2}$ and translation 3 units down. Correct order is:

1 Horizontal translation 1 unit left

2 Vertical dilation of scale factor $\dfrac{1}{2}$

3 Vertical translation 3 units down

Because vertical transformations can be done first, the order 2–3–1 is also possible.

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b For $y = 3(2x - 6)^2 + 1$:

First put the equation in the form $y = kf(a(x + b)) + c$.

$$y = 3(2x - 6)^2 + 1$$
$$= 3[2(x - 3)]^2 + 1$$

Horizontal transformations: dilation $a = 2$ and translation $b = -3$.

Vertical transformations: dilation $k = 3$ and translation $c = 1$.

1 Horizontal dilation of scale factor $\frac{1}{2}$

2 Horizontal translation 3 units right

3 Vertical dilation of scale factor 3

4 Vertical translation 1 unit up

The order 3–4–1–2 is also possible.

Alternative method: There is another possible order, if you notice that $y = 3(2x - 6)^2 + 1$ is of the form $y = kf(ax + b) + c$, where the $(ax + b)$ is not factorised, so we can do the horizontal translation first, then horizontal dilation.

The horizontal translation is 6 units right ($b = -6$) followed by a horizontal dilation of scale factor $\frac{1}{2}$, then 3 and 4 as above.

c We require $y = kf(a(x + b)) + c$.

Horizontal transformations: dilation $a = -1$ and translation $b = -4$.

Vertical transformations: dilation $k = 5$ and translation $c = -9$.

Horizontal transformations: $y = kf(-1(x - 4)) + c$

Add vertical transformations: $y = 5f(-(x - 4)) - 9$

This answer can also be written as $y = 5f(-x + 4) - 9$ or $y = 5f(4 - x) - 9$.

Domain and range

We can find the domain and range of functions without drawing their graphs.

Effect of transformations on domain and range

Horizontal transformations change x values so affect the domain.

Vertical transformations change y values so affect the range.

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EXAMPLE 15

Find the domain and range of:

a $f(x) = -3(x-2)^2 + 5$

b $y = 5\sqrt{2x+1}$

Solution

a $y = x^2$ has domain $(-\infty, \infty)$ and range $[0, \infty)$.

Horizontal transformations affect the domain:

No horizontal dilation.

Horizontal translation 2 units right: domain of $x - 2$ is $(-\infty, \infty)$ so domain of $f(x)$ is unchanged.

Vertical transformations affect the range:

Vertical dilation, scale factor -3: Range of y is $[0, \infty)$, so range of $3y$ is 3 times as much, so no change for $[0, \infty)$.

But the $-$ sign in -3 means the y is reflected in the x-axis, so range of $-3y$ is $(-\infty, 0]$.

Vertical translation 5 units up: Range of $-3y$ is $(-\infty, 0]$ so range of $-3y + 5$ is $(-\infty, 5]$.

So $y = -3(x-2)^2 + 5$ has domain $(-\infty, \infty)$ and range $(-\infty, 5]$.

b $y = \sqrt{x}$ has domain $[0, \infty)$ and range $[0, \infty)$.

Horizontal transformations affect the domain:

Domain of $2x + 1$ is $[0, \infty)$ so $2x + 1 \geq 0$.

$2x \geq -1$

$x \geq -\dfrac{1}{2}$

Vertical transformations affect the range:

Vertical dilation, scale factor 5: Range of $\sqrt{2x+1}$ is $[0, \infty)$, so range of $5\sqrt{2x+1}$ is 5 times as much, so unchanged.

No vertical translation.

So $y = 5\sqrt{2x+1}$ has domain $\left[-\dfrac{1}{2}, \infty\right)$ and range $[0, \infty)$.

Exercise 2.05 Combinations of transformations

1 The point (2, –6) lies on the function $y = f(x)$. Find the coordinates of its image if the function is:

a horizontally translated 3 units to the right and vertically translated 5 units down

b translated 4 units up and 3 units to the left

c translated 7 units to the right and 9 units up

d translated 11 units down and 4 units to the left

2 Find the equation of the transformed function where $f(x) = x^5$ is reflected:

a in the x-axis and vertically dilated with scale factor 4

b in the y-axis and horizontally dilated with scale factor 3

3 Find the equation of each transformed function.

a $y = x^3$ is translated 3 units down and 4 units to the left

b $f(x) = |x|$ is translated 9 units up and 1 unit to the right

c $f(x) = x$ is dilated vertically with a scale factor of 3 and translated down 6 units

d $y = e^x$ is reflected in the x-axis and translated up 2 units

e $y = x^3$ is horizontally dilated by a scale factor of $\frac{1}{2}$ and translated down 5 units

f $f(x) = \frac{1}{x}$ is vertically dilated by a factor of 2 and horizontally dilated by a factor of 3

g $f(x) = \sqrt{x}$ is reflected in the y-axis, vertically dilated by a scale factor of 3 and horizontally dilated by a scale factor of $\frac{1}{2}$

h $y = \ln x$ is horizontally dilated by a scale factor of 3 and translated upwards by 2 units

i $f(x) = \log_2 x$ is horizontally dilated by a scale factor of $\frac{1}{4}$ and vertically dilated by a scale factor of 3

j $y = x^2$ is horizontally dilated by a scale factor of 2 and translated down 3 units

4 Describe the transformations to $y = x^3$ in the correct order if the transformed function has equation:

a $y = (x - 1)^3 + 7$

b $y = 4x^3 - 1$

c $y = -5x^3 - 3$

d $y = 2(x + 7)^3$

e $y = 6(2x - 4)^3 + 5$

f $y = 2(3x + 9)^3 - 10$

5 Describe the transformations in their correct order for each of the functions from:

a $y = \log x$ to $y = 2 \log (x + 3) - 1$

b $f(x) = x^2$ to $f(x) = -(3x)^2 + 9$

c $y = e^x$ to $y = 2e^{5x} - 3$

d $f(x) = \sqrt{x}$ to $f(x) = 4\sqrt{x - 7} + 1$

e $y = |x|$ to $y = |-2(x + 1)| - 1$

f $y = \frac{1}{x}$ to $y = -\frac{1}{2x} + 8$

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6 The point $(8, -12)$ lies on the function $y = f(x)$. Find the coordinates of the image point when the function is transformed into:

a $y = 3f(x - 1) + 5$ **b** $y = -f(2x) - 7$ **c** $y = 2f(x + 3) - 1$

d $y = 6f(-x) + 5$ **e** $y = -2f(2x - 4) - 3$

7 Given the function $y = f(x)$, find the coordinates of the image of (x, y) if the function is:

a translated 6 units down and 3 units to the right

b reflected in the y-axis and translated 6 units up

c vertically dilated with scale factor 2 and translated 5 units to the left

d horizontally dilated with scale factor 3 and translated 5 units up

e reflected in the x-axis, vertically dilated with scale factor 8, translated 6 units to the left, horizontally dilated with scale factor 5 and translated 1 unit down

8 Find the equation of the transformed function if $y = f(x)$ is:

a translated 2 units down and 1 unit to the left

b translated 5 units to the right and 3 units up

c reflected in the x-axis and translated 4 units to the right

d reflected in the y-axis and translated up 2 units

e reflected in the x-axis and horizontally dilated with a factor of 4

f vertically dilated by a scale factor of 2 and translated 2 units down

9 Find the equation of the transformed function using the correct order of transformations for $y = kf(a(x + b)) + c$.

a $f(x) = \frac{1}{x}$ is reflected in the y-axis, translated up 3 units and dilated vertically by a scale factor of 9

b $y = x^2$ is translated down by 6 units and by 2 units to the left and is horizontally dilated with scale factor $\frac{1}{5}$

c $f(x) = \ln x$ has a vertical dilation with factor 8, a vertical translation of 3 down, a horizontal dilation with factor 2 and a horizontal translation of 5 to the right

d $y = \sqrt{x}$ has a vertical translation of 4 up, a horizontal translation of 4 to the left, a reflection in the y-axis and a vertical dilation with factor 9

e $f(x) = |x|$ is translated up by 7 units, dilated horizontally by a factor of $\frac{1}{6}$ and reflected in the x-axis

f $y = x^3$ is translated 4 units to the left then dilated horizontally with scale factor $\frac{1}{4}$

g $y = 2^x$ is translated up by 5 units, translated 2 units to the right, then is vertically dilated with scale factor 6

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10 Find the domain and range of each function.

a $f(x) = (x+3)^2 + 5$ **b** $y = 5\,|-2x| - 2$ **c** $f(x) = \dfrac{1}{2x-4} + 1$

d $y = 4^{3x} + 2$ **e** $f(x) = 3\log(3x-6) - 5$

11 **a** By completing the square, write the equation for the parabola $y = x^2 + 2x - 7$ in the form $y = (x+a)^2 + b$.

b Describe the transformations on $y = x^2$ that result in the function $y = x^2 + 2x - 7$.

12 Describe the transformations that change $y = x^2$ into the function $y = x^2 - 10x - 3$.

13 The function $y = f(x)$ is transformed to the function $y = kf(a(x+b)) + c$.
Find the coordinates of the image point of (x, y) when:

a $c = 5, b = -3, k = 2$ and $a = \dfrac{1}{2}$

b $c = -2, b = 6, k = -1$ and $a = 3$

14 **a** Find the equation of the transformed graph if $x^2 + y^2 = 9$ is translated 3 units to the right and 4 units up.

b The circle $x^2 + y^2 = 1$ is transformed into the circle $x^2 - 4x + y^2 + 6y + 12 = 0$. Describe how the circle is transformed.

WS Graphing transformed functions

2.06 Graphs of functions with combined transformations

We can find points and sketch the graphs of functions that are changed by a combination of transformations. Translations are the easiest transformations to use since they shift the graph while keeping it the same size and shape.

Translations of functions

EXAMPLE 16

Sketch the graph of $y = (x-2)^2 - 5$.

Solution

$y = (x-2)^2 - 5$ is transformed from $y = x^2$ by a horizontal translation of 2 units to the right and a vertical translation of 5 units down.

The vertex (turning point) of parabola $y = x^2$ is $(0, 0)$.

So the vertex of $y = (x-2)^2 - 5$ is $(0+2, 0-5) \equiv (2, -5)$.

Sketching the graph, we keep the shape of $y = x^2$ and shift it to the new vertex.

We can find the intercepts for a more accurate graph.

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For x-intercepts, $y = 0$:

$$0 = (x - 2)^2 - 5$$
$$5 = (x - 2)^2$$
$$\pm\sqrt{5} = x - 2$$
$$2 \pm \sqrt{5} = x$$

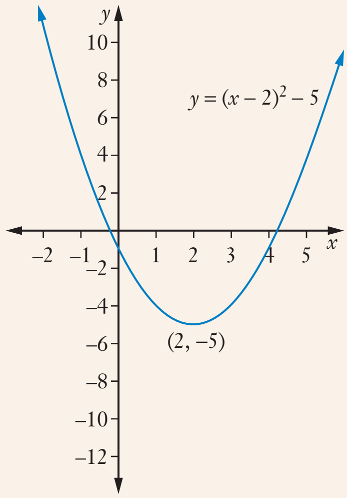
For y-intercepts, $x = 0$:

$$y = (0 - 2)^2 - 5$$
$$= 4 - 5$$
$$= -1$$

So the x-intercepts are approximately 4.2, −0.2.

To find other points on the graph, you can transform points on $y = x^2$ the same way as for the vertex.

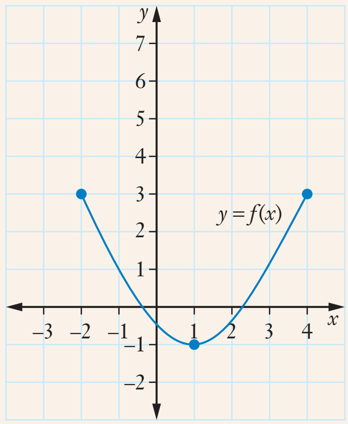
Sketch the graph using a scale on each axis that will show the information. For example, the vertex is at (2, −5) so the y values must go down as far as $y = -5$.



EXAMPLE 17

The graph $y = f(x)$ shown is reflected in the y-axis, dilated vertically with a scale factor of 2 and translated 1 unit up.

Sketch the graph of the transformed function.



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Solution

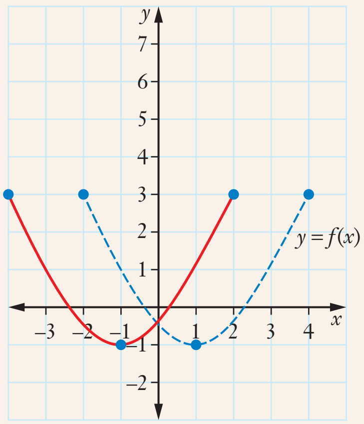
A reflection in the y-axis is a horizontal dilation with scale factor -1.

Multiply each x value by -1.

$x = 1$ becomes $x = -1$

$x = 4$ becomes $x = -4$

$x = -2$ becomes $x = 2$

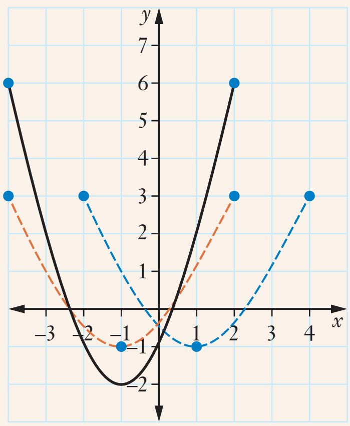


For a vertical dilation with scale factor 2:

Multiply each y value by 2.

$y = -1$ becomes $y = -2$

$y = 3$ becomes $y = 6$

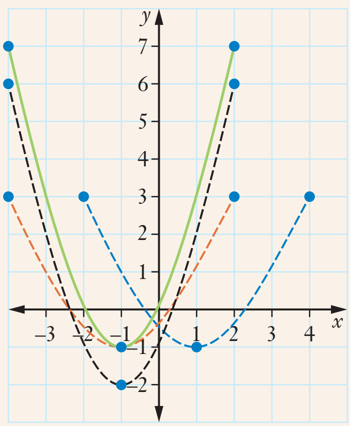


For a vertical translation 1 unit up:

Add 1 to y values.

$y = 6$ becomes $y = 7$

$y = -2$ becomes $y = -1$



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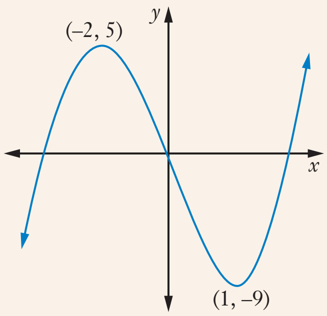
In the previous example, we took one transformation at a time. In the next example, we take transformations together (in the correct order) and plot images of key points on the original (parent) curve.

EXAMPLE 18

a The function $y = f(x)$ is sketched below with stationary (turning) points as shown.



i Describe the transformations if $y = f(x)$ is transformed to $y = 3f(x + 1) - 2$ and how they change the coordinates (x, y) of the parent function.

ii Find the coordinates of the image of each stationary point when the function is transformed.

iii Sketch the graph of $y = 3f(x + 1) - 2$.

b **i** Describe the transformations if $y = |x|$ is transformed to $y = -\left|\frac{x}{2}\right| + 3$ and the image of point (x, y) on the parent function.

ii Sketch the transformed function.

Solution

a **i** Transformations (in order) are:

A horizontal translation 1 unit to the left:

So (x, y) becomes $(x - 1, y)$.

A vertical dilation, scale factor 3:

So $(x - 1, y)$ becomes $(x - 1, 3y)$.

A vertical translation 2 units down:

So $(x - 1, 3y)$ becomes $(x - 1, 3y - 2)$.

ii For $(-2, 5)$:

Image becomes $(-2 - 1, 3 \times 5 - 2) \equiv (-3, 13)$.

For $(1, -9)$:

Image becomes $(1 - 1, 3 \times [-9] - 2) \equiv (0, -29)$.

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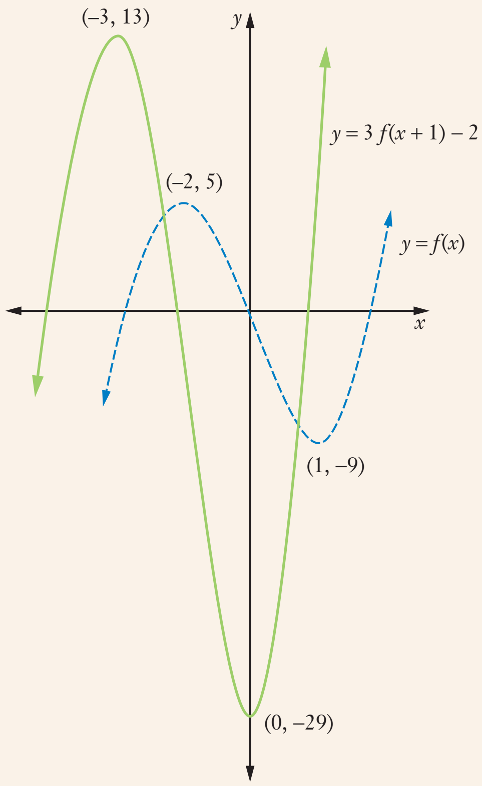
iii $y = f(x)$ passes through (0, 0).

Image becomes $(0 - 1, 3 \times 0 - 2)$
$\equiv (-1, -2)$

Sketch the graph showing this information using a suitable scale on each axis. For example, the y values must go up to 13 and down to -29.



b **i** Transformations (in order) are:

A horizontal dilation, scale factor 2:

So (x, y) becomes $(2x, y)$.

A vertical dilation, scale factor -1 (reflection in the x-axis):

So $(2x, y)$ becomes $(2x, -y)$.

A vertical translation 3 units up:

So $(2x, -y)$ becomes $(2x, -y + 3)$.

ii The intercepts of $y = |x|$ are at (0, 0).

Image of (0, 0) is $(2 \times 0, -0 + 3) \equiv (0, 3)$.

We can find the intercepts on $y = -\left|\frac{x}{2}\right| + 3$

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For x-intercept, $y = 0$:

$$0 = -\left|\frac{x}{2}\right| + 3$$

$$-3 = -\left|\frac{x}{2}\right|$$

$$3 = \left|\frac{x}{2}\right|$$

$$\pm 3 = \frac{x}{2}$$

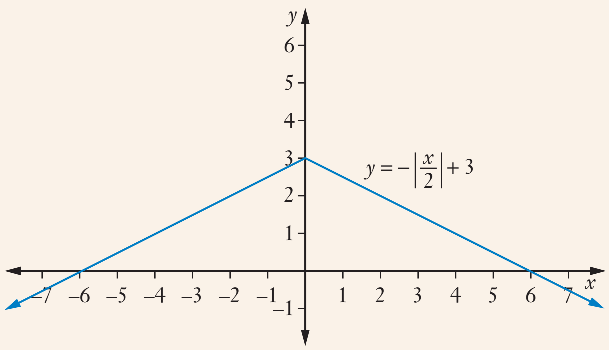
$$\pm 6 = x$$

For y-intercept, $x = 0$:

$$y = -\left|\frac{0}{2}\right| + 3$$

$$= 3$$

Sketching this information using an appropriate scale gives the graph.



Exercise 2.06 Graphs of functions with combined transformations

1 Given $f(x) = x^2$, sketch the graph of:

a $f(x) = x^2 + c$ when

i $c > 0$ **ii** $c < 0$

b $f(x) = (x + b)^2$ when

i $b > 0$ **ii** $b < 0$

c $f(x) = kx^2$ when

i $k > 1$ **ii** $0 < k < 1$ **iii** $k = -1$

d $f(x) = (ax)^2$ when

i $a > 1$ **ii** $0 < a < 1$ **iii** $a = -1$

2 Sketch the graph of the transformed function if the parabola $y = x^2$ is transformed into:

a $y = (x + 2)^2 + 4$ **b** $y = (x - 3)^2 - 1$ **c** $y = (x - 1)^2 + 3$

d $y = -(x + 1)^2 - 2$ **e** $y = 2(x - 1)^2 - 4$

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3 Sketch the graph of the transformed function if the cubic function $y = x^3$ is transformed into:

a $y = (x - 1)^3 + 2$ **b** $y = (x - 2)^3 - 3$ **c** $y = -(x + 1)^3 + 4$

d $y = 2(x + 3)^3 - 5$ **e** $y = 3(x - 1)^3 - 2$

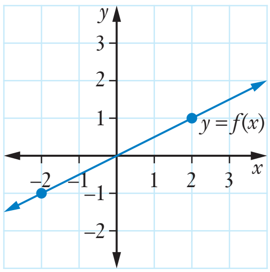
4 A cubic function has stationary points at (6, 1) and (−3, −2).

a Find the images of these points if the function is transformed to $y = -2f(3x) + 1$.

b Sketch the graph of the transformed function.

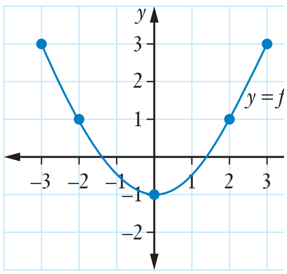
5 Given each function $y = f(x)$, sketch the graph of the transformed function.

a



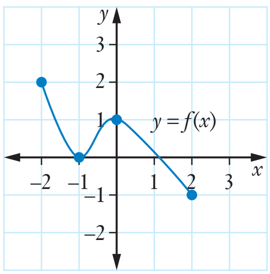
i $y = 3f(x - 1)$ **ii** $y = -f(2x) + 3$

b



i $y = 3f(x + 3) - 2$ **ii** $y = -2f\left(\frac{x}{4}\right) + 3$

c



i $y = 2f(-x) - 1$ **ii** $y = -3f(2x + 4) + 2$

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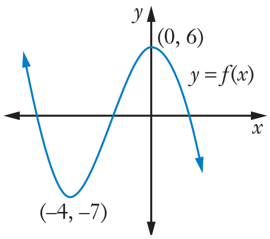
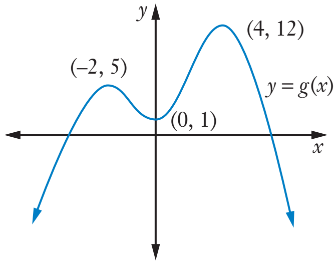
6 For the function $y = f(x)$ with turning points as shown, sketch the transformed function if it is vertically dilated with scale factor 3, translated 4 units down, and horizontally translated 2 units to the left.



7 For the function $y = g(x)$ with turning points as shown, sketch the graph of the transformed function $y = -g[2(x - 1)] - 5$.



8 Sketch the graph of:

a $y = -3(x - 2)^3 + 1$

b $y = 2e^{x+1} - 4$

c $f(x) = 3\sqrt{x - 2} - 1$

d $y = 2\,|\,3x\,| + 4$

e $y = -(3x)^2 + 1$

9 Sketch the graph of:

a $y = 3 - 2\ln x$

b $f(x) = -2e^x + 1$

c $y = 1 - (x + 1)^3$

d $y = \dfrac{2}{x - 1} + 3$

e $y = -2(x - 3) + 1$

10 a The coordinates of the image of (x, y) when $y = f(x)$ is transformed to $y = 3f(x - 2) + 1$ are $(-3, 2)$. Find the original point (x, y).

b Sketch the graph of the original function $y = f(x)$ if $y = 3f(x - 2) + 1$ is a cubic function with turning points $(-3, 2)$ and $(2, -4)$.

11 The coordinates of the image point of the vertex (x, y) of a parabola are $(-24, 18)$ when $y = f(x)$ is transformed as shown below. Find the coordinates of the original point (x, y) and sketch the graph of the original quadratic function.

a $y = 3f(x - 2) - 5$

b $y = -5f[3(x + 1)]$

c $y = 2f(2x - 6) - 3$

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2.07 Equations and inequalities

Solving inequalities graphically

We can use the graphs of transformed functions to solve equations.

EXAMPLE 19

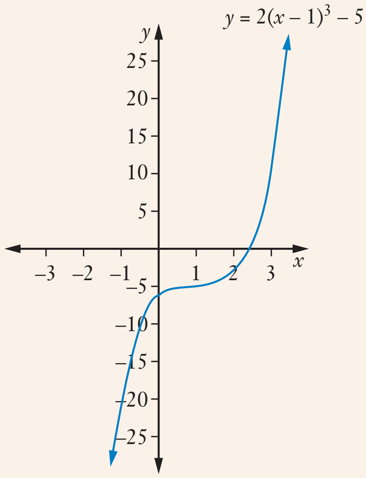
The graph of the cubic function $y = 2(x - 1)^3 - 5$ is shown.

a Solve graphically:

i $2(x - 1)^3 - 5 = 0$

ii $2(x - 1)^3 - 5 = 10$

b Solve each of the equations in part **a** algebraically.



Solution

a **i** The solution of $2(x - 1)^3 - 5 = 0$ is where $y = 0$ (x-intercepts).

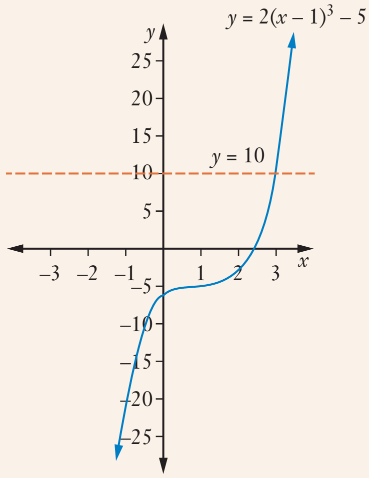
From the graph, the x-intercept is 2.4.

The solution is $x = 2.4$.

ii Draw the line $y = 10$ on the graph.

The solution of $2(x - 1)^3 - 5 = 10$ is where the line intersects the graph.

The solution is $x = 2.9$.



b **i**

$$\begin{aligned} 2(x-1)^3 - 5 &= 0 \\ 2(x-1)^3 &= 5 \\ (x-1)^3 &= 2.5 \\ x - 1 &= \sqrt[3]{2.5} \\ x &= \sqrt[3]{2.5} + 1 \\ &= 2.36 \end{aligned}$$

ii

$$\begin{aligned} 2(x-1)^3 - 5 &= 10 \\ 2(x-1)^3 &= 15 \\ (x-1)^3 &= 7.5 \\ x - 1 &= \sqrt[3]{7.5} \\ x &= \sqrt[3]{7.5} + 1 \\ &= 2.96 \end{aligned}$$

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We can use transformed functions to find solutions to practical questions.

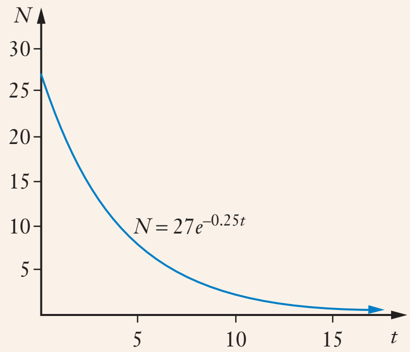
EXAMPLE 20

The graph of $N = 27e^{-0.25t}$ shows the number N of cases of measles over t weeks in a country region.

a Use the graph to find the solution to $27e^{-0.25t} = 10$.

b State the meaning of this solution.

c Solve the equation algebraically.

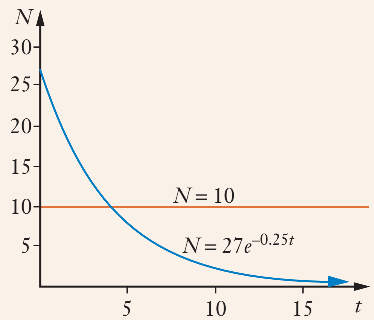


Solution

a Draw the line $N = 10$ on the graph.

The solution will be where the line intersects the graph.

The solution is $t = 4$.



b This solution means that after 4 weeks there will be 10 cases of measles.

c

$$\begin{aligned}
27e^{-0.25t} &= 10 \\
e^{-0.25t} &= \frac{10}{27} \\
\ln e^{-0.25t} &= \ln \frac{10}{27} \\
-0.25t &= -0.99325\ \ldots \\
t &= \frac{-0.99325\ \ldots}{-0.25} \\
&= 3.97300\ldots \\
&\approx 3.97
\end{aligned}$$

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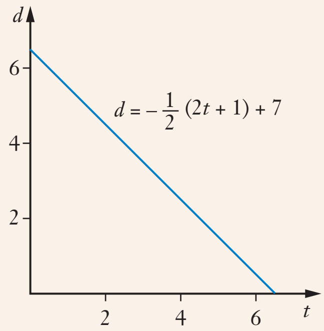
We can solve inequalities graphically.

EXAMPLE 21

a The graph is of the function

$d = -\frac{1}{2}(2t + 1) + 7$ where d is the distance (in cm) of a marble at t seconds as it rolls towards a barrier.



Solve graphically and explain the solutions:

i $-\frac{1}{2}(2t + 1) + 7 = 4$

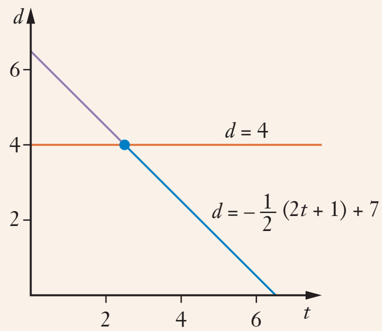
ii $-\frac{1}{2}(2t + 1) + 7 \geq 4$

b Sketch the graph of $y = 2(x + 3)^2 - 5$ and solve graphically:

i $2(x + 3)^2 - 5 = 3$ **ii** $2(x + 3)^2 - 5 < 3$

Solution

a Draw the line $d = 4$ across the graph.



i From the graph, the solution of $-\frac{1}{2}(2t + 1) + 7 = 4$ is $x = 2.5$.

This means that at 2.5 seconds, the marble is 4 cm from the barrier.

ii The solution of $-\frac{1}{2}(2t + 1) + 7 \geq 4$ is all the t values on and above the line $d = 4$, shown in purple.

For this part of the graph, $t \leq 2.5$.

Because $t \geq 0$ (time is never negative), $0 \leq t \leq 2.5$ is the solution.

This means that for the first 2.5 seconds the marble is 4 cm or more from the barrier.

b The function $y = 2(x + 3)^2 - 5$ is a transformation of $y = x^2$.

The vertex of $y = x^2$ is (0, 0).

The image of (0, 0) is $(0 - 3, 0 \times 2 - 5) \equiv (-3, -5)$

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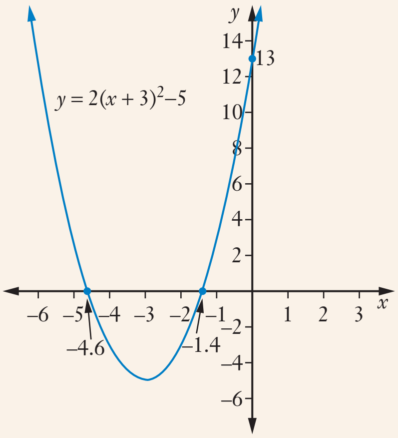
For x-intercept, $y = 0$:

$$0 = 2(x+3)^2 - 5$$
$$5 = 2(x+3)^2$$
$$2.5 = (x+3)^2$$
$$\pm\sqrt{2.5} = x + 3$$
$$\pm\sqrt{2.5} - 3 = x$$
$$-1.4, -4.6 = x$$

For y-intercept, $x = 0$:

$$y = 2(0+3)^2 - 5$$
$$= 2(9) - 5$$
$$= 13$$

Sketch the graph using a suitable scale on the axes.

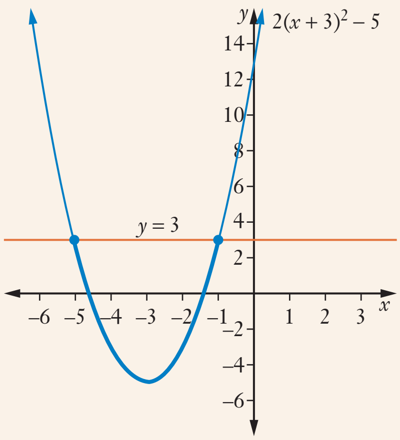


i Draw the line $y = 3$.

From the graph, the solution of $2(x+3)^2 - 5 = 3$ is $x = -5, -1$.

ii The solution of $2(x+3)^2 - 5 < 3$ is all x values below the line $y = 3$.

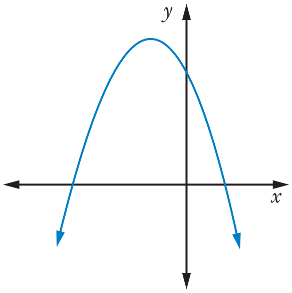
From the graph, the solution of $2(x+3)^2 - 5 < 3$ is $-5 < x < -1$.



Exercise 2.07 Equations and inequalities

1 For each function $y = f(x)$, state how many solutions there are for the equation $f(x) = 0$.

a



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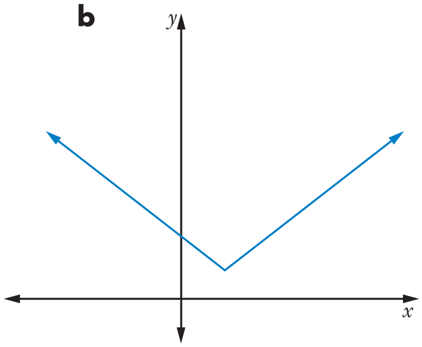

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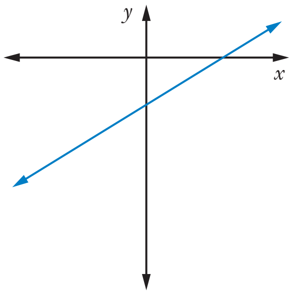

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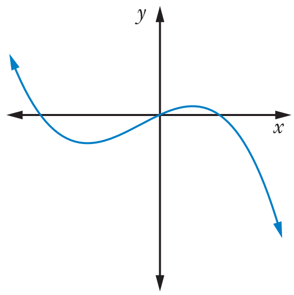

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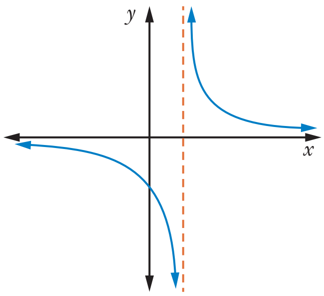

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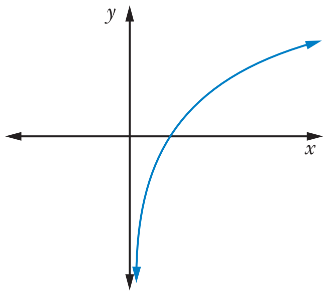

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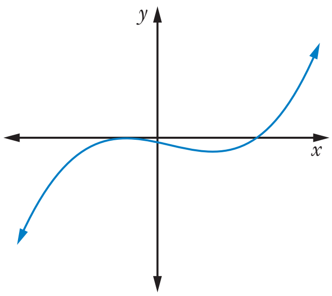

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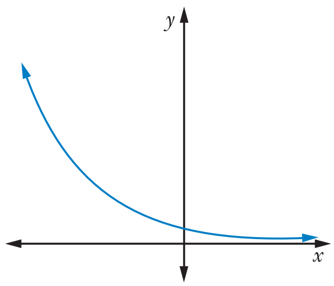

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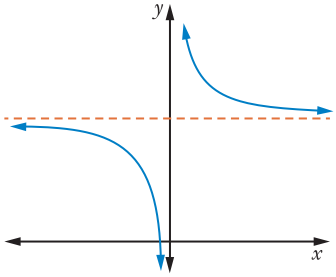

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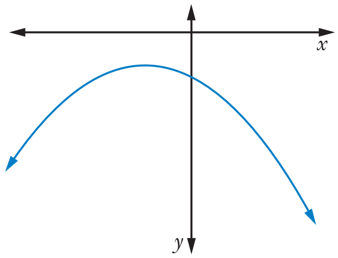


2 The graph of the quadratic function $f(x) = -2(x + 1)^2 + 3$ is shown.

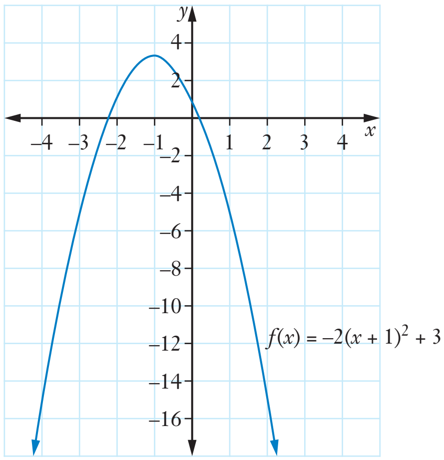
a Solve graphically:

i $-2(x + 1)^2 + 3 = 1$

ii $-2(x + 1)^2 + 3 = -2$

iii $-2(x + 1)^2 + 3 = 0$

b Solve $-2(x + 1)^2 + 3 = 0$ algebraically.



3 The graph of the linear function $f(x) = 3(4x - 5) - 2$ is shown.
Use the graph to solve:

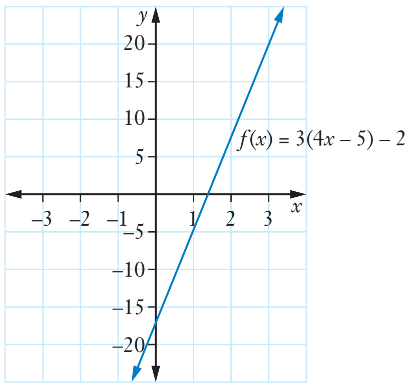
a $3(4x - 5) - 2 = 0$

b $3(4x - 5) - 2 = 5$

c $3(4x - 5) - 2 = -15$

d $3(4x - 5) - 2 > 10$

e $3(4x - 5) - 2 \leq 20$



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4 **a** Sketch the graph of the cubic function $y = -(x+3)^3 + 1$.

b Solve graphically:

i $-(x+3)^3 + 1 = 0$

ii $-(x+3)^3 + 1 = -10$

iii $-(x+3)^3 + 1 = -20$

c Solve $-(x+3)^3 + 1 = 0$ algebraically.

5 **a** Sketch the graph of $y = 3|x-2| + 4$.

b How many solutions does the equation $3|x-2| + 4 = 1$ have?

c Solve $3|x-2| + 4 = 10$ graphically and check your solutions algebraically.

6 **a** Sketch the graph of the function $f(x) = \dfrac{2}{x-3} - 4$, showing asymptotes.

b Solve the equation $\dfrac{2}{x-3} - 4 = -5$.

c Solve $\dfrac{2}{x-3} - 4 = -2$.

7 The formula for the area of a garden with side x metres is given by $A = -3(x-2)^2 + 18$.

a Draw the graph of the area of the garden.

b From the graph, solve the equation $-3(x-2)^2 + 18 = 10$.

8 A factory has costs according to the formula $C = 2(x+1)^2 + 3$, where C stands for costs in \$1000s and x is the number of products made.

a Draw the graph of the costs.

b Find the factory overhead (cost when no products are made).

c Solve $2(x+1)^2 + 3 = 20$ from the graph and explain your answer.

9 Loudness in decibels (dB) is given by $\text{dB} = 10\log\left(\dfrac{x}{I}\right)$ where I is a constant.

a Sketch the graph of the function given $I = 2$.

b From the graph solve the equation:

i $10\log\left(\dfrac{x}{I}\right) = 5$

ii $10\log\left(\dfrac{x}{I}\right) = 2$

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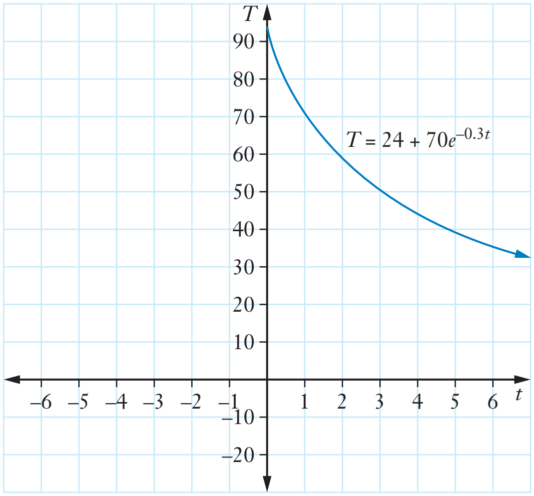
10 According to Newton's law of cooling, the temperature T of an object as it cools over time t minutes is given by the formula $T = A + Be^{-kt}$. The graph shown is for the formula $T = 24 + 70e^{-0.3t}$ for a metal ball that has been heated and is now cooling down.



a From the graph, solve these equations and explain what the solutions mean.

i $24 + 70e^{-0.3t} = 50$ **ii** $24 + 70e^{-0.3t} = 30$

b Solve these equations algebraically:

i $24 + 70e^{-0.3t} = 80$ **ii** $24 + 70e^{-0.3t} = 26$

c What temperature will the object approach as t becomes large?

Can you give a reason for this?

11 a Sketch the graph of $y = (x - 1)^2 - 2$.

b From the graph, solve:

i $(x - 1)^2 - 2 = 2$ **ii** $(x - 1)^2 - 2 \geq 2$ **iii** $(x - 1)^2 - 2 < 2$

12 a Sketch the graph of $f(x) = -(2x + 4)^2 + 1$.

b From the graph, solve:

i $-(2x + 4)^2 + 1 = -3$ **ii** $-(2x + 4)^2 + 1 > -3$ **iii** $-(2x + 4)^2 + 1 \leq -3$

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2. TEST YOURSELF

Practice quiz

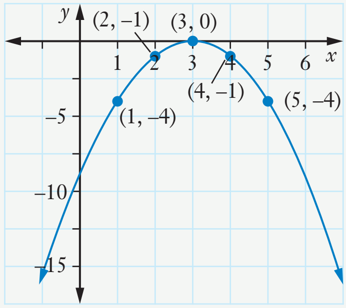
For Questions **1** to **3**, choose the correct answer **A**, **B**, **C** or **D**.

1 The function $y = f(x)$ transformed to $y = f(x - 8)$ is:

A a vertical translation 8 units up
B a horizontal translation 8 units to the right
C a vertical translation 8 units down
D a horizontal translation 8 units to the left

2 The graph below is a transformation of $y = x^2$. Find its equation.

A $y = (-x + 3)^2$ **B** $y = (-x - 3)^2$
C $y = -(x + 3)^2$ **D** $y = -(x - 3)^2$



3 Find the coordinates of the image of (x, y) when the function $y = f(x)$ is transformed to $y = -2f(x + 1) + 4$.

A $(x + 1, -2y - 4)$ **B** $(x + 1, -2y + 4)$
C $(x - 1, -2y + 4)$ **D** $(-x + 1, 2y + 4)$

4 **a** Draw the graph of $y = e^{x-1} - 2$.
b Use the graph to solve $e^{x-1} - 2 = 8$.
c Solve $e^{x-1} - 2 = 20$ algebraically.

5 The point $(24, 36)$ lies on the graph of $y = f(x)$. Find the coordinates of its image point if the function is transformed to:

a $y = 3f(4x) - 1$ **b** $y = f[3(x + 2)] + 4$ **c** $y = 5f(-x) - 3$
d $y = -2f(x + 7) - 3$ **e** $y = -f(2x - 8) + 5$

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6 Find the equation of each transformed function.

a $y = x^3$ is translated:

i 3 units up **ii** 7 units to the left

b $y = |x|$ is dilated:

i vertically with scale factor 3

ii horizontally with scale factor 2

c $f(x) = \ln x$ is dilated vertically with factor 5 and reflected in the y-axis.

d $f(x) = \dfrac{1}{x}$ is reflected in the x-axis and translated 4 units to the right.

e $f(x) = 3^x$ is dilated vertically with scale factor 9, dilated horizontally with scale factor $\dfrac{1}{3}$ and translated 6 units down and 2 units to the right.

7 a State the meaning of the constants a, b, c and k in the function $y = kf(a(x + b)) + c$ and the effect they have on the graph of the function $y = f(x)$.

b Describe the effect on the graph of the function if:

i $k = -1$ **ii** $a = -1$

8 Show that if $y = x^2$ is dilated vertically with scale factor 3, reflected in the x-axis and translated 1 unit up, the transformed function is even.

9 a Draw the graph of $y = 2(x - 3) + 5$.

b From the graph, solve:

i $2(x - 3) + 5 \leq 7$ **ii** $2(x - 3) + 5 > 9$

10 The population of a city over time t years is given by $P = 2e^{0.4(t + 1)}$ where P is population in 10 000s.

a Sketch the graph of the population.

b Use the graph to solve $2e^{0.4(t + 1)} = 5$, and explain the meaning of the solution.

11 Find the equation of the transformed function if $f(x) = x^4$ is horizontally translated 4 units to the left.

12 If (8, 2) lies on the graph of $y = f(x)$, find the coordinates of the image of this point when the function is transformed to $y = -4f[2(x + 1)] - 3$.

13 Solve both graphically and algebraically:

a $2(3x - 6)^2 - 5 = 9$ **b** EXT1 $2(3x - 6)^2 - 5 > 9$ **c** EXT1 $2(3x - 6)^2 - 5 \leq 9$

14 The function $y = f(x)$ is transformed to $y = -7f(x - 3) - 4$.

a Find the coordinates of the image of (x, y).

b If the image point is $(-3, 3)$, find the value of x and y.

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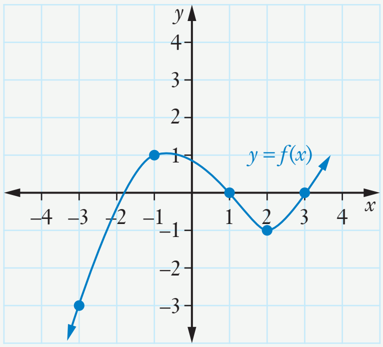
15 From the graph of $y = f(x)$ shown, draw the graph of:

a $y = 2f(x - 1)$

b $y = -f(x) - 2$



16 By drawing the graph of $y = 2(x + 1)^2 - 8$, solve:

a $2(x + 1)^2 - 8 \le 0$

b $2(x + 1)^2 - 8 > 0$

17 Sketch on the same set of axes:

a $y = x^2$ and $y = -4x^2 + 3$

b $y = |x|$ and $y = -|x - 1| + 2$

c $f(x) = e^x$ and $f(x) = \dfrac{e^{x+2}}{2} - 1$

d $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x+2} + 1$

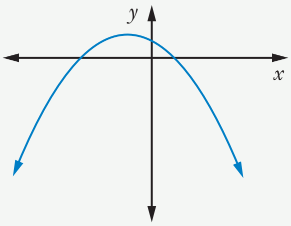
e $y = x^3$ and $y = 2(x - 3)^3 + 1$

f $f(x) = \ln x$ and $f(x) = \ln(-x) + 5$

g $y = \sqrt{x}$ and $y = 2\sqrt{x + 4} - 1$

18 Find the number of solutions of $f(x) = 0$ given the graph of each function $y = f(x)$.

a



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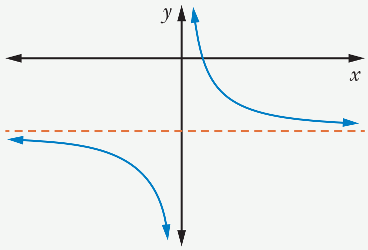

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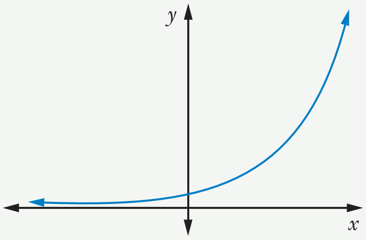

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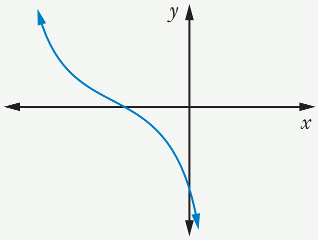

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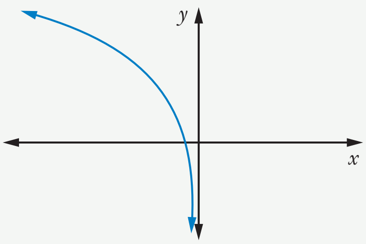

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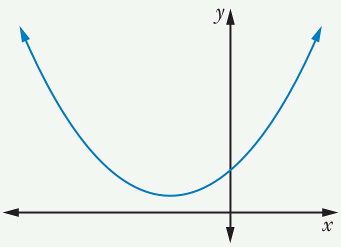

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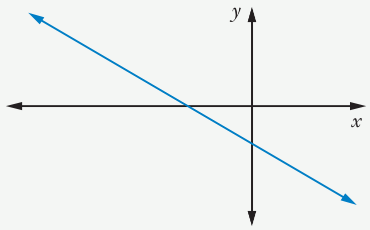

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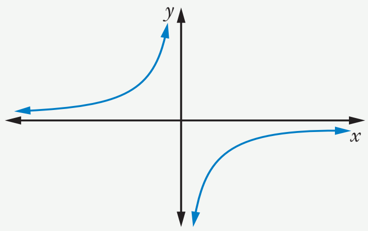

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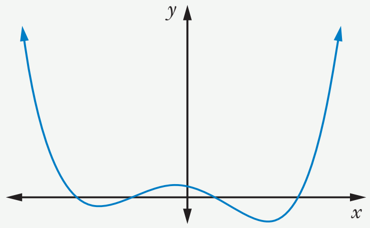

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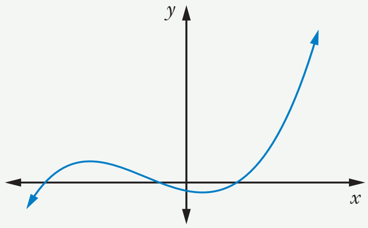

19 **a** Show that $x^2 + y^2 = r^2$ is not a function and describe its graph.

b Find 2 functions that together form $x^2 + y^2 = r^2$.

c By applying a vertical dilation with scale factor a to both these functions, what shape does the combination of these stretched functions make?

20 The point (x, y) lies on the function $y = f(x)$. The image of (x, y) is the point $(12, 6)$ when the function is transformed to $y = -6f(2x + 8)$. Find the coordinates of (x, y).

21 **a** Draw the graph of $y = (x - 2)^2 + 1$.

b From the graph, solve:

i $(x - 2)^2 + 1 = 10$ **ii** $(x - 2)^2 + 1 > 10$ **iii** $(x - 2)^2 + 1 \le 10$

22 Point (x, y) lies on $y = f(x)$. Find the image of (x, y) if the function is transformed to:

a $y = 3f(x + 1) - 5$ **b** $y = -2f[2(x - 6)] + 4$

c $y = 5f(-x) - 3$ **d** $y = -3f(-3x + 9) - 1$

23 State whether the function $y = f(x)$ is stretched or compressed if it is dilated:

a vertically with scale factor 7 **b** horizontally with scale factor $\frac{1}{6}$

c horizontally with scale factor 3 **d** vertically with scale factor $\frac{1}{4}$

e horizontally with scale factor $\frac{7}{6}$

24 Find the domain and range of:

a $y = 3(x - 7)^2 - 10$ **b** $y = -|x + 1| + 2$ **c** $y = -\dfrac{2}{x - 3} - 5$

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2. CHALLENGE EXERCISE

1 A ball is thrown into the air from a height of 1 m, reaches its maximum height of 3 m after 1 second and after 2 seconds it is 1 m high.

a The path of the ball follows the shape of a parabola. Find the equation of the height h of the ball over time t seconds.

b After how long does the ball fall to the ground?

c Put the function in the form $h = kf[a(t + b)] + c$ and describe the transformations to change $h = t^2$ into this equation.

2 a If (4, –3) lies on the function $y = f(x)$, find the coordinates of its image point.

i P on $y = 3f(x + 3) + 1$

ii Q on $y = -f(2x) - 3$

iii R on $y = f(2x - 2) + 1$

b Find the equation of the linear function passing through P that is perpendicular to QR.

c If $y = x$ is transformed into this linear function, describe the transformations.

3 a Show that $\dfrac{2x-7}{x-3} = -\dfrac{1}{x-3} + 2$.

b Sketch the graph of $y = \dfrac{2x-7}{x-3}$ and state its domain and range.

c Solve:

i $\dfrac{2x-7}{x-3} \geq 0$ **ii** $\dfrac{2x-7}{x-3} < 2$

4 a If $y = \dfrac{1}{x}$ is dilated horizontally with scale factor 2, explain why the equation of the transformed function is the same as if it was dilated vertically with scale factor 2.

b Is this the same result for the function $y = \dfrac{1}{x^2}$? Why?

5 a What is the equation of the axis of symmetry of the quadratic function $f(x) = ax^2 + bx + c$?

b What types of transformations on this function will change the axis of symmetry?

c Find the equation of the axis of symmetry of the quadratic function:

i $f(x) = 2(x + 1)^2 - 2$

ii $y = -(x - 3)^2 + 7$

iii $y = k(x + b)^2 + c$

iv $y = k(ax + b)^2 + c$

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6 The function $y = \sin x$ in the domain $[0, 2\pi]$ is transformed by a reflection in the x-axis, a vertical dilation scale factor 3, a horizontal dilation scale factor 2 and a vertical translation 1 unit down.

- **a** Find the equation of the transformed function.
- **b** State the amplitude, period and centre of the transformed function.

7 EXT1 For the function $f(x) = x(x - 1)(x + 2)$, sketch:

- **a** $y^2 = f(x)$
- **b** $y^2 = 2f(x) - 3$

8 The circle $x^2 + 4x + y^2 - 6y + 12 = 0$ is transformed by a vertical translation 3 units down and a horizontal translation 5 units right. Find the equation of the transformed circle.

9 The function $y = 2^x$ is transformed to $y = 3(2^{-3x-6}) - 5$. Describe the transformations applied to the function.

10 The polynomial $P(x) = x^3 - 3x - 2$ is translated up 2 units and then reflected in the y-axis. Find the equation of the transposed polynomial.